QUANTITATIVE RANK DISTRIBUTION CONJECTURE OVER $\mathbb{F}_q(t)$

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ABSTRACT. We combine the exact counting of all elliptic curves over $K = \mathbb{F}_q(t)$ with $\mathrm{char}(K) > 3$ by Bejleri, Satriano and the author, together with the torsion-free nature of most elliptic curves over global function fields proven by Phillips, and the overarching conjecture of Goldfeld and Katz-Sarnak regarding the *"Distribution of Ranks of Elliptic Curves"*. Consequently, we arrive at the quantitative statement which naturally renders even finer conjecture regarding the lower order main terms differing for the number of E/K with $|E(K)| = 1$ and $E(K) = \mathbb{Z}$.

1. INTRODUCTION

The study of fibrations of curves and abelian varieties over a smooth algebraic curve lies at the heart of the classification theory of algebraic surfaces and rational points on varieties over global fields such as the field $\mathbb{Q}$ of rational numbers or the field $\mathbb{F}_q(t)$ of rational functions over the finite field $\mathbb{F}_q$.

For the case of elliptic curves over global fields of positive characteristic, we know the sharp enumeration of elliptic curves over $K = \mathbb{F}_q(t)$ with $\mathrm{char}(\mathbb{F}_q) > 3$ by considering the totality of rational points on the fine modular curve $\overline{\mathcal{M}}_{1,1}$ over K through the height moduli framework of Bejleri, Satriano and the author [BPS22].

Specifically, recall that the height of the discriminant of an elliptic curve E over K is given by $ht(\Delta) := q^{\deg \Delta} = q^{12n}$ for some integer $n \in \mathbb{Z}_{\geq 0}$ (also called the Faltings height of E). We count globally minimal Weierstrass models over K. In order to keep track of the primitive roots of unity contained in $\mathbb{F}_q$, we define the following auxiliary function

$$\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.1 (Theorem 9.7 of [BPS22]). *Let $n \in \mathbb{Z}_{\geq 0}$ and $\mathrm{char}(\mathbb{F}_q) > 3$. The counting function $\mathcal{N}^{min}\big(\mathbb{F}_q(t),\ B\big)$, which counts the number of $\mathbb{F}_q$–isomorphism classes of minimal elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ ordered by the multiplicative height of the discriminant $ht(\Delta) = q^{12n} \leq B$, is given by the following.*

$$\begin{aligned} \mathcal{N}^{min}(\mathbb{F}_q(t),\ B) = 2\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6} - 2B^{1/6} \\ + \delta(6)\cdot 4\left(\frac{q^5-1}{q^5-q^4}\right)B^{1/2} + \delta(4)\cdot 2\left(\frac{q^3-1}{q^3-q^2}\right)B^{1/3} \\ + \delta(6)\cdot 4 + \delta(4)\cdot 2 \end{aligned}$$

The lower order main term of order $B^{1/6}$ comes from subtracting the μ_2 twist families of generically singular $j = \infty$ isotrivial elliptic curves. And the lower order main *terms of order $B^{1/2}$ and $B^{1/3}$ respectively come from counting the μ_6 and μ_4 twist families of isotrivial elliptic curves having strictly additive bad reductions with extra automorphisms concentrated at the special j-invariants $j = 0$ and $j = 1728$.*

While fundamental on its own, the above exact count is consequential as it provides the exact denominator for the question of proportion over all E/K. In this regard, note that the set of rational points $E(K)$ is a finitely generated abelian group by Mordell-Weil i.e. $E(K) = \mathbb{Z}^r \oplus T$ with algebraic rank $r \in \mathbb{Z}_{\geq 0}$ and torsion subgroup T. Here, we recall another fundamental fact that *'Most elliptic curves over global function fields are torsion free'* proven by Phillips [Phi22] similar to [Duk97].

Theorem 1.2 (Theorem 2.2 of [Phi22]). *The set of torsion-free elliptic curves over global function fields has density 1.*

'Unboundedness' is settled for $E/\mathbb{F}_q(t)$ as follows.

Theorem 1.3 (Tate-Shafarevich & Ulmer). *Ranks of non-constant elliptic curves over $\mathbb{F}_q(t)$ are unbounded (in both the isotrivial* [JTT67] *and non-isotrivial cases* [Ulm02]*).*

Lastly, we have the overarching conjecture of *"Distribution of Ranks of Elliptic Curves"* by the classical works of Goldfeld [Gol79] and Katz-Sarnak [KS99].

Conjecture 1.4 (Goldfeld and Katz-Sarnak Conjecture). Over any number field, 50% of all elliptic curves (when ordered by height) have Mordell-Weil rank $r = 0$ and the other 50% have Mordell-Weil rank $r = 1$. Moreover, higher Mordell-Weil ranks $r \geq 2$ constitute 0% of all elliptic curves, even though there may exist infinitely many such elliptic curves.

Classical works of [Bru92, BS15a, BS15b] proved the upper bounds for average rank of $E/\mathbb{Q}$ and [dJ02] for $E/\mathbb{F}_q(t)$.

2. Borromean rings

Combining the above 2 Theorems and 1 Conjecture, if assumed to be true, leads us to the following *Quantitative Rank Distribution Conjecture over $K = \mathbb{F}_q(t)$.*

Conjecture 2.1. Let $n \in \mathbb{Z}_{\geq 0}$ and $\mathrm{char}(\mathbb{F}_q) > 3$. The counting function $\mathcal{N}^r_T(\mathbb{F}_q(t),\, B)$, which counts the number of $\mathbb{F}_q$–isomorphism classes of minimal elliptic curves over $\mathbb{P}^1_{\mathbb{F}_q}$ with algebraic rank $r \in \mathbb{Z}_{\geq 0}$ and torsion subgroup T ordered by the multiplicative height of the discriminant $ht(\Delta) = q^{12n} \leq B$, is given by the following.

$$\mathcal{N}^{r=0}_{T=0}(\mathbb{F}_q(t),\ B) = \left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6} + o(B^{\frac{5}{6}}),$$

$$\mathcal{N}^{r=1}_{T=0}(\mathbb{F}_q(t),\ B) = \left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6} + o(B^{\frac{5}{6}}),$$

$$\mathcal{N}^{r\geq 2}_{T}(\mathbb{F}_q(t),\ B) = o(B^{\frac{5}{6}}),\ \text{where all o are little-o.}$$

Specifically, $|E(K)| = 1$ and $E(K) = \mathbb{Z}$ each corresponds to 50% of all elliptic curves over K ordered by discriminant height having *equal* main leading term $B^{5/6}$ with *identical* leading coefficient $\left(\frac{q^9-1}{q^8-q^7}\right)$. Additionally,

$$\mathcal{N}^{r=0}_{T=0}(\mathbb{F}_q(t),\ B) \neq \mathcal{N}^{r=1}_{T=0}(\mathbb{F}_q(t),\ B)$$

Namely, the exact counting formulas for $\mathcal{N}^{r=0}_{T=0}(\mathbb{F}_q(t),\ B)$ and $\mathcal{N}^{r=1}_{T=0}(\mathbb{F}_q(t),\ B)$ do not coincide as respective counting functions have distinct lower order main terms.

Remark 2.2. It is intriguing that we cannot reach this level of fine resolution without all 3 statements working together. Namely, if we only knew Theorem 1.2 and Conjecture 1.4 then we conclude at most that $|E(K)| = 1$ and $E(K) = \mathbb{Z}$ each corresponds to 50% of all E/K without having the precise main leading term with its leading coefficient. On the other hand, if we only knew Theorem 1.1 and Conjecture 1.4 then we conclude at most that $\mathcal{N}^{r=0}_{T}(\mathbb{F}_q(t),\ B)$ and $\mathcal{N}^{r=1}_{T}(\mathbb{F}_q(t),\ B)$ have the main leading term of $\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6}$ without being able to pin down the precise Mordell-Weil group structure either as $|E(K)| = 1$ or $E(K) = \mathbb{Z}$ respectively as there are various possibilities for torsion subgroup T (c.f. [Maz77, CP80, McD18]). Finally, if we only knew Theorem 1.1 and Theorem 1.2 then we conclude at most that the main leading term $2\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6}$ of all E/K somehow distributes to torsion-free E/K with *unspecified* algebraic rank $r \in \mathbb{Z}_{\geq 0}$. In this regard, Conjecture 1.4 gives arithmetic geometric meaning to main leading term with its leading coefficient.

Remark 2.3. For the higher rank $r \geq 2$ elliptic curves over K, it is impossible to make any finer conjecture than $\mathcal{N}^{r\geq 2}_{T}(\mathbb{F}_q(t),\ B) = o(B^{\frac{5}{6}})$ at this point. Similarly for the rest of the cases, $\mathcal{N}^{r=1}_{T\neq 0}(\mathbb{F}_q(t),\ B) = o(B^{\frac{5}{6}})$ and $\mathcal{N}^{r=0}_{T\neq 0}(\mathbb{F}_q(t),\ B) = o(B^{\frac{5}{6}})$.

Remark 2.4. It is unlikely that $\mathcal{N}^{r=0}_{T=0}(\mathbb{F}_q(t),\ B)$ and $\mathcal{N}^{r=1}_{T=0}(\mathbb{F}_q(t),\ B)$ coincide i.e. either they both have no lower order terms or lower order main terms of equal orders and identical coefficients. While ambitious, it is an interesting question nonetheless to prove or disprove either way.

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