ℓ-ADIC ÉTALE COHOMOLOGY OF THE MODULI OF GENERALIZED ELLIPTIC CURVES WITH PRESCRIBED LEVEL STRUCTURE

JUN–YONG PARK

ABSTRACT. We determine ℓ-adic étale cohomology with Frobenius weights for the moduli stack $\mathcal{L}_{1,12n} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ of stable elliptic fibrations over $\mathbb{P}^1$ with a marked Weierstrass section over $\overline{\mathbb{F}}_q$ with $\mathrm{char}(\overline{\mathbb{F}}_q) \neq 2, 3$. Also for the moduli stacks $\mathcal{L}_{1,12n}^{[\Gamma]} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma]$-level structure over $\overline{\mathbb{F}}_q$ with $\mathrm{char}(\overline{\mathbb{F}}_q) \neq 2$ for $[\Gamma] = [\Gamma_1(2)], [\Gamma_1(4)], [\Gamma(2)]$ and $\neq 3$ for $[\Gamma] = [\Gamma_1(3)]$.

1. Introduction

Fix a basefield K, and define for $a, b \in \mathbb{N}$ the weighted projective stack $\mathcal{P}(a,b) := [(\mathbb{A}^2_{x,y} \setminus 0)/\mathbb{G}_m]$ where $\lambda \in \mathbb{G}_m$ acts by $\lambda \cdot (x,y) = (\lambda^a x, \lambda^b y)$. In this paper we find ℓ-adic étale cohomology and eigenvalues of geometric Frobenius map for the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$, parameterizing degree $n \in \mathbb{N}$ morphisms $f : \mathbb{P}^1 \to \mathcal{P}(a,b)$. The moduli $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ was formulated in [HP] when the characteristic of base field K is not dividing a or b. We recall the *arithmetic aspect* of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ as follows:

Theorem 1.1 (Theorem 1 of [HP]). *If* $\mathrm{char}(\mathbb{F}_q)$ *does not divide* a *or* b*, then the weighted point count* $\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)))$ *over* $\mathbb{F}_q$ *is*

$$\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) = q^{(a+b)n+1} - q^{(a+b)n-1}$$

By ℓ-adic Leray spectral sequence with respect to faithfully flat evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)) \to \mathcal{P}(a,b)$ (see Proposition 2.13), we compute ℓ-adic étale Betti numbers $\dim_{\mathbb{Q}_\ell}\left(H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)\right)$ showing $\mathbb{Q}_\ell$-vector spaces are all one dimensional for $i = 0, 3$ and vanishes for all other i implying that the ℓ-adic rational cohomology type of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ is a 3-sphere. Consequently, we have the Main Theorem through the Grothendieck-Lefschetz trace formula (see Theorem 2.10) showing the *geometric aspect* of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ as follows:

Theorem 1.2. *Hom stack* $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ *with* $n \in \mathbb{N}$ *over* $\overline{\mathbb{F}}_q$ *with* $\mathrm{char}(\overline{\mathbb{F}}_q)$ *not dividing* a *or* b *for* $a, b \in \mathbb{N}$ *has the following compactly supported* ℓ*-adic étale cohomology and Galois representations of mixed Tate type*

$$H^i_{\acute{e}t,c}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-((a+b)n+1)) & i = 2(a+b)n+2, \\ \mathbb{Q}_\ell(-((a+b)n-1)) & i = 2(a+b)n-1, \\ 0 & \textit{else.} \end{cases}$$

By the Poincaré duality, the ordinary ℓ-adic étale cohomology is equal to

$$H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(0) & i = 0, \\ \mathbb{Q}_\ell(-2) & i = 3, \\ 0 & \textit{else.} \end{cases}$$

We describe the three special cases of interest, One is $\mathcal{P}(1,1) \cong \mathbb{P}^1$ which gives the space $\mathrm{Hom}_n(\mathbb{P}^1, \mathbb{P}^1)$ of degree $n \in \mathbb{N}$ unbased rational maps between projective lines studied in depth by [Segal, CCMM, Silverman, KS].

Corollary 1.3. *The space* $\mathrm{Hom}_n(\mathbb{P}^1, \mathbb{P}^1)$ *with* $n \in \mathbb{N}$ *unbased rational maps* $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ *over* $\overline{\mathbb{F}}_q$ *of any cardinality* q *has the following compactly supported ℓ-adic étale cohomology and Galois representations of mixed Tate type*

$$H^i_{\acute{e}t,c}(\mathrm{Hom}_n(\mathbb{P}^1, \mathbb{P}^1)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-(2n+1)) & i = 4n+2, \\ \mathbb{Q}_\ell(-(2n-1)) & i = 4n-1, \\ 0 & \textit{else.} \end{cases}$$

The second is $\mathcal{P}(4,6) \cong \overline{\mathcal{M}}_{1,1}$ isomorphic to the proper Deligne–Mumford stack of stable elliptic curves over base field K with $\mathrm{char}(K) \neq 2,3$ with the coarse moduli space $\overline{M}_{1,1} \cong \mathbb{P}^1$ parameterizing the j–invariants. This gives us the Hom stack $\mathcal{L}_{1,12n} := \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(4,6) \cong \overline{\mathcal{M}}_{1,1})$ which is isomorphic to the moduli stack of stable elliptic fibrations over $\mathbb{P}^1$ formulated in [HP].

Corollary 1.4. *Hom stack* $\mathcal{L}_{1,12n} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ *with* $n \in \mathbb{N}$ *over* $\overline{\mathbb{F}}_q$ *with* $\mathrm{char}(\overline{\mathbb{F}}_q) \neq 2,3$ *isomorphic to the moduli stack of stable elliptic fibrations over* $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$ *with* $12n$ *nodal singular fibers and a marked Weierstrass section has the following compactly supported ℓ-adic étale cohomology and Galois representations of mixed Tate type*

$$H^i_{\acute{e}t,c}(\mathcal{L}_{1,12n/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-(10n+1)) & i = 20n+2, \\ \mathbb{Q}_\ell(-(10n-1)) & i = 20n-1, \\ 0 & \textit{else.} \end{cases}$$

The third is $\mathcal{P}(a,b) \cong \overline{\mathcal{M}}_{1,1}[\Gamma]$ with the compactified modular curve isomorphic to the proper Deligne–Mumford stack of generalized elliptic curves with $[\Gamma]$-level structure (where Γ is $\Gamma(n)$ or $\Gamma_1(n)$) (see Proposition 2.4). Recall that a level structure $[\Gamma_1(n)]$ on an elliptic curve E is a choice of point $P \in E$ of exact order n in the smooth part of E such that over every geometric point of the base scheme every irreducible component of E contains a multiple of P (see [KM, §1.4]). And a level structure $[\Gamma(2)]$ on an elliptic curve E is a choice of isomorphism $\phi : \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to E(2)$ where $E(2)$ is the scheme of 2-torsion Weierstrass points (i.e., kernel of the multiplication-by-2 map $[2] : E \to E$) (see [DR, II.1.18 & IV.2.3]). This gives us the Hom stack $\mathcal{L}^{[\Gamma]}_{1,12n} := \mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b) \cong \overline{\mathcal{M}}_{1,1}[\Gamma])$ which is isomorphic to the moduli stack of generalized elliptic curves over $\mathbb{P}^1$ with $[\Gamma]$-level structure formulated in the Appendix of [HP2].

Corollary 1.5. *Hom stack* $\mathcal{L}^{[\Gamma]}_{1,12n} := \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}[\Gamma])$ *with* $n \in \mathbb{N}$ *over* $\overline{\mathbb{F}}_q$ *with* $\mathrm{char}(\overline{\mathbb{F}}_q) \neq 2$ *for* $[\Gamma] = [\Gamma_1(2)], [\Gamma_1(4)], [\Gamma(2)]$ *and* $\neq 3$ *for* $[\Gamma] = [\Gamma_1(3)]$ *isomorphic to the moduli stack of generalized elliptic curves over* $\mathbb{P}^1_{\overline{\mathbb{F}}_q}$ *with* $12n$ *nodal singular fibers and* $[\Gamma]$*-level structure has the following compactly supported* ℓ*-adic étale cohomology and Galois representations of mixed Tate type*

$$H^i_{\acute{e}t,c}(\mathcal{L}^{[\Gamma_1(2)]}_{1,12n/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-(6n+1)) & i = 12n+2, \\ \mathbb{Q}_\ell(-(6n-1)) & i = 12n-1, \\ 0 & \textit{else.} \end{cases}$$

$$H^i_{\acute{e}t,c}(\mathcal{L}^{[\Gamma_1(3)]}_{1,12n/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-(4n+1)) & i = 8n+2, \\ \mathbb{Q}_\ell(-(4n-1)) & i = 8n-1, \\ 0 & \textit{else.} \end{cases}$$

$$H^i_{\acute{e}t,c}(\mathcal{L}^{[\Gamma_1(4)]}_{1,12n/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-(3n+1)) & i = 6n+2, \\ \mathbb{Q}_\ell(-(3n-1)) & i = 6n-1, \\ 0 & \textit{else.} \end{cases}$$

$$H^i_{\acute{e}t,c}(\mathcal{L}^{[\Gamma(2)]}_{1,12n/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-(4n+1)) & i = 8n+2, \\ \mathbb{Q}_\ell(-(4n-1)) & i = 8n-1, \\ 0 & \textit{else.} \end{cases}$$

Lastly, we consider $\mathrm{Z}(\mathfrak{X}, t)$ the *zeta function* of a smooth Deligne–Mumford stack $\mathfrak{X}$ of finite type over $\mathbb{F}_q$ with its weighted $\mathbb{F}_q$-point count $\#_q(\mathfrak{X})$.

Definition 1.6 (Definition 3.2.3 of [Behrend])**.** Let $\mathfrak{X}$ be an algebraic stack of finite type over $\mathbb{F}_q$. We define the zeta function $\mathrm{Z}(\mathfrak{X}, t) \in \mathbb{Q}[[t]]$, where t is a complex variable, as a formal power series:

$$\mathrm{Z}(\mathfrak{X}, t) := \exp\left(\sum_{\nu=1}^{\infty} \frac{t^\nu}{\nu} \cdot \#_{q^\nu}(\mathfrak{X})\right)$$

Knowing the weighted $\mathbb{F}_q$–point count $\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) = q^{(a+b)n+1} - q^{(a+b)n-1}$ by [HP, Theorem 1], we can write down $\mathrm{Z}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)), t)$ as a formal power series. The following theorem shows that $\mathrm{Z}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)), t)$ is in fact a rational function.

Theorem 1.7 (Theorem 3.2.5 of [Behrend])**.** *Let $\mathfrak{X}$ be a smooth Deligne-Mumford stack of finite type over $\mathbb{F}_q$. Then $\mathrm{Z}(\mathfrak{X}, t)$ is rational and we have*

$$\mathrm{Z}(\mathfrak{X}, t) = \prod_{i=0}^{2dim(\mathfrak{X})} \det\left(1 - \mathrm{Frob}_q^* \ \cdot t \mid H^i_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)\right)^{(-1)^{(i+1)}}$$

We have the zeta function of the moduli stack as a rational function.

Corollary 1.8. *The zeta function $\mathrm{Z}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)), t)$ of the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ parameterizing degree n morphisms $f : \mathbb{P}^1 \to \mathcal{P}(a,b)$ over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \nmid a, b$ is equal to*

$$\mathrm{Z}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)), t) = \frac{\left(1 - q^{(a+b)n-1} \cdot t\right)}{\left(1 - q^{(a+b)n+1} \cdot t\right)}$$

2. Étale cohomology with eigenvalues of geometric Frobenius

We first recall some definitions concerning the stack $\mathcal{P}(a,b)$.

Definition 2.1. For $a, b \in \mathbb{N}$, the weighted projective stack $\mathcal{P}(a,b)$ is defined as the quotient stack

$$\mathcal{P}(a,b) := [(\mathbb{A}^2_{x,y} \setminus 0)/\mathbb{G}_m],$$

where $\lambda \in \mathbb{G}_m$ acts by $\lambda \cdot (x,y) = (\lambda^a x, \lambda^b y)$. In this case, x and y have degrees a and b respectively. The line bundle $\mathcal{O}_{\mathcal{P}(a,b)}(m)$ is defined to be the line bundle associated to the sheaf of degree m homogeneous rational functions without poles on $\mathbb{A}^2_{x,y} \setminus 0$. We say that a map $\mathbb{P}^1 \to \mathcal{P}(a,b)$ has degree n if $f^*\mathcal{O}_{\mathcal{P}(a,b)}(1) \cong \mathcal{O}_{\mathbb{P}^1}(n)$.

Note that $\mathcal{P}(a,b)$ is not an (effective) orbifold when $\gcd(a,b) \neq 1$. In this case, the cyclic isotropy group scheme $\mu_{\gcd(a,b)}$ is the generic stabilizer of $\mathcal{P}(a,b)$. Nevertheless, the following proposition shows that it behaves well in most characteristics as a tame Deligne–Mumford stack:

Proposition 2.2. *The 1-dimensional weighted projective stack $\mathcal{P}(a,b)$ is a tame Deligne–Mumford stack over a basefield K if $\mathrm{char}(K)$ does not divide a or b.*

Proof. For any algebraically closed field extension $\overline{K}$ of K, any point $y \in \mathcal{P}(a,b)(\overline{K})$ is represented by the coordinates $(y_0, y_1) \in \mathbb{A}^2_{\overline{K}}$ with its stabilizer group as the subgroup of $\mathbb{G}_m$ fixing (y_0, y_1). Hence, any stabilizer group of such $\overline{K}$-points is $\mathbb{Z}/u\mathbb{Z}$ where u divides a or b. Since the characteristic of K does not divide the orders of $\mathbb{Z}/\lambda_i\mathbb{Z}$ for any i, the stabilizer group of y is $\overline{K}$-linearly reductive. Hence, $\mathcal{P}(a,b)$ is tame by [AOV, Theorem 3.2]. Note that the stabilizer groups constitute fibers of the diagonal $\Delta : \mathcal{P}(a,b) \to \mathcal{P}(a,b) \times_K \mathcal{P}(a,b)$. Since $\mathcal{P}(a,b)$ is of finite type and $\mathbb{Z}/u\mathbb{Z}$'s are unramified over K whenever u does not divide a or b, Δ is unramified as well. Therefore, $\mathcal{P}(a,b)$ is also Deligne–Mumford by [Olsson2, Theorem 8.3.3]. ∎

The tameness is analogous to flatness for stacks in positive/mixed characteristic as it is preserved under base change by [AOV, Corollary 3.4]. Moreover, if a stack $\mathcal{X}$ is tame and Deligne–Mumford, then the formation of the coarse moduli space $c : \mathcal{X} \to X$ commutes with base change as well by [AOV, Corollary 3.3].

Example 2.3. When $\mathrm{char}(K) \neq 2, 3$, [Hassett, Proposition 3.6] shows that one example is given by the proper Deligne–Mumford stack of stable elliptic curves $(\overline{\mathcal{M}}_{1,1})_K \cong [(\mathrm{Spec}\ K[a_4, a_6] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(4,6)$ by using the short Weierstrass equation $y^2 = x^3 + a_4 x + a_6 x$, where $\lambda \cdot a_i = \lambda^i a_i$ for $\lambda \in \mathbb{G}_m$ and $i = 4, 6$. This is no longer true when $\mathrm{char}(K) \in \{2, 3\}$, as the Weierstrass equations are more complicated.

Roughly, a generalized elliptic curve X over $\mathbb{P}^1_K$ can be thought of as a flat family of semistable elliptic curves admitting a group structure, such that a finite group scheme $\mathcal{G} \to \mathbb{P}^1_K$ (determined by Γ) embeds into X and its image meets every irreducible component of every geometric fibers of X. By the work of Deligne and Rapoport [DR] (summarized in [Conrad, §2] and also in [Niles, §2]), we consider the generalized elliptic curves over $\mathbb{P}^1_K$ with $[\Gamma]$–structures (where Γ is $\Gamma(n)$ or $\Gamma_1(n)$) over a field K (focusing on $K = \mathbb{F}_q$).

Proposition 2.4. *The moduli stack $\overline{\mathcal{M}}_{1,1}[\Gamma]$ of generalized elliptic curves with $[\Gamma]$-level structure is isomorphic to the following when over a field K:*

(1) *if $\mathrm{char}(K) \neq 2$, the tame Deligne–Mumford moduli stack of generalized elliptic curves with $[\Gamma_1(2)]$-level structure is isomorphic to*
$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)])_K \cong [(\mathit{Spec}\ K[a_2, a_4] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(2,4),$$
(2) *if $\mathrm{char}(K) \neq 3$, the tame Deligne–Mumford moduli stack of generalized elliptic curves with $[\Gamma_1(3)]$-level structure is isomorphic to*
$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)])_K \cong [(\mathit{Spec}\ K[a_1, a_3] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(1,3),$$
(3) *if $\mathrm{char}(K) \neq 2$, the tame Deligne–Mumford moduli stack of generalized elliptic curves with $[\Gamma_1(4)]$-level structure is isomorphic to*
$$(\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)])_K \cong [(\mathit{Spec}\ K[a_1, a_2] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(1,2),$$

(4) if $\mathrm{char}(K) \neq 2$*, the tame Deligne–Mumford moduli stack of generalized elliptic curves with* $[\Gamma(2)]$*-level structure is isomorphic to*

$$(\overline{\mathcal{M}}_{1,1}[\Gamma(2)])_K \cong [(Spec\ K[a_2, a_2] - (0,0))/\mathbb{G}_m] = \mathcal{P}_K(2,2),$$

where $\lambda \cdot a_i = \lambda^i a_i$ *for* $\lambda \in \mathbb{G}_m$ *and* $i = 1,2,3,4$*. Thus, the* a_i*'s have degree* i *respectively.*

Proof. Proof of the first, second, third and fourth equivalence can be found in [Behrens, §1.3], [HMe, Proposition 4.5], [Meier, Examples 2.1] and [Stojanoska, Proposition 7.1] respectively. By Proposition 2.2, the weighted projective stacks are tame Deligne–Mumford as well. ∎

Definition 2.5. The stack $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ is defined to be the Hom stack of degree n morphisms $\mathbb{P}^1 \to \mathcal{P}(a,b)$. Concretely,

$$\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)) = [T/\mathbb{G}_m]$$

where $T \subset (H^0(\mathcal{O}_{\mathbb{P}^1}(an))\backslash 0) \oplus (H^0(\mathcal{O}_{\mathbb{P}^1}(bn))\backslash 0)$ is the open subset of pairs of nonzero polynomials $(\widetilde{u}(z,t), \widetilde{v}(z,t))$, homogenous of degrees a and b respectively, sharing no common factor, and $\lambda \in \mathbb{G}_m$ acts by $\lambda \cdot (\widetilde{u}(z,t), \widetilde{v}(z,t)) := (\widetilde{u}(\lambda z, \lambda t), \widetilde{v}(\lambda z, \lambda t)) = (\lambda^a \cdot \widetilde{u}(z,t), \lambda^b \cdot \widetilde{v}(z,t))$.

Proposition 2.6. *For* $a, b, n \in \mathbb{N}$ *and* $\mathrm{char}(K) \nmid a, b$*, the Hom stack* $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ *parameterizing degree* n *morphisms* $f : \mathbb{P}^1 \to \mathcal{P}(a,b)$ *is a smooth separated tame Deligne–Mumford stack of finite type and dimension* $(a+b)n+1$*.*

Proof. This was established in [HP, Proposition 9, Proof of Theorem 1]. To recall the major points therein, $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b)) = [T/\mathbb{G}_m]$ is a smooth Deligne–Mumford stack by [Olsson, Theorem 1.1], admitting T as a smooth schematic cover. As $\mathbb{G}_m$ acts on T properly with positive weights $a, b \in \mathbb{N}$ the quotient stack $[T/\mathbb{G}_m]$ is separated. It is tame by [AOV, Theorem 3.2] since $\mathrm{char}(K) \nmid a, b$. ∎

As a polynomial in $H^0(\mathcal{O}_{\mathbb{P}^1}(an)) \setminus 0$ or $H^0(\mathcal{O}_{\mathbb{P}^1}(bn)) \setminus 0$ is determined by its restriction away from $\infty = [0:1]$ (i.e., by evaluating $t = 1$), we may consider T as the set of not-necessarily-monic pairs of nonzero coprime polynomials $(u(z), v(z))$ such that either $\deg u = an$ and $0 \le \deg v \le bn$ or $\deg v = bn$ and $0 \le \deg u \le an$. Here the degree conditions are precisely ensuring that $\widetilde{u}, \widetilde{v}$ do not share a common zero at ∞.

Definition 2.7. For any $k, \ell \ge 0$, define

$$\mathrm{Poly}_1^{k,l} := \{(u(z), v(z)) : u(z), v(z) \text{ are monic and coprime of degrees } k, \ell \text{ respectively}\}.$$

Equivalently, $\mathrm{Poly}_1^{(k,l)}$ is the complement of the resultant hypersurface $\mathcal{R}^{(k,\ell)} \subset \mathrm{Sym}^k \mathbb{A}^1 \times \mathrm{Sym}^\ell \mathbb{A}^1 = \mathbb{A}^k \times \mathbb{A}^\ell$ of pairs of monic polynomials $(u(z), v(z))$ of degrees k, ℓ respectively which share a factor.

The arithmetic of $\mathrm{Poly}_1^{(k,l)}$ was studied in [HP], inspired by the work of [FW, Segal]. As the resultant hypersurface $\mathcal{R}^{(k,\ell)}$ is defined over $\mathbb{Z}$, it makes sense to define $\mathrm{Poly}_1^{k,\ell} = (\mathbb{A}^k \times \mathbb{A}^\ell) \setminus \mathcal{R}^{(k,\ell)}$ as a variety over $\mathbb{Z}$.

Proposition 2.8 (Proposition 18 of [HP]). *Fix $d_1, d_2 \geq 0$. Then for any prime power q :*

$$|\mathrm{Poly}_1^{(d_1,d_2)}(\mathbb{F}_q)| = \begin{cases} q^{d_1+d_2} - q^{d_1+d_2-1}, & \text{if } d_1, d_2 > 0 \\ q^{d_1+d_2}, & \text{if } d_1 = 0 \text{ or } d_2 = 0 \end{cases}$$

We recall that the cohomology in torsion-free coefficient coincide for the fine moduli stack and its coarse moduli space. We show this for the étale cohomology over basefield $\overline{\mathbb{F}}_q$ in $\mathbb{Q}_\ell$-coefficient by following the proof of the [Sun, Proposition 7.3.2].

Lemma 2.9. *Let $\mathfrak{X}$ be a smooth separated tame Deligne–Mumford stack of finite type over $\overline{\mathbb{F}}_q$ and the coarse moduli map $c : \mathfrak{X} \to X$ giving the coarse moduli space X. Then for all i, the pullback map*

$$c^* : H^i_{\acute{e}t}(X_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^i_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$$

is an isomorphism.

Proof. As $\mathfrak{X}$ is a smooth separated tame Deligne–Mumford stack of finite type over $\overline{\mathbb{F}}_q$, we can cover X by étale charts U such that pull-back of U in $\mathfrak{X}$ is the quotient stack of an algebraic space by a finite group [AOV, Theorem 3.2.]. The lemma follows from the ℓ-adic Leray spectral sequence as in [Behrend, Theorem 1.2.5] once we have shown that the canonical map $\mathbb{Q}_\ell \to \mathrm{R}c_*\mathbb{Q}_\ell$ is an isomorphism. It suffices to show the isomorphism étale locally on X and hence we assume $\mathfrak{X} = [V/G]$ for some algebraic space V under the action of finite group G where $\mathrm{char}(\overline{\mathbb{F}}_q)$ does not divide $|G|$. Let $q : V \to \mathfrak{X}$ be the canonical morphism. Observe that we have $\mathbb{Q}_\ell \simeq (q_*\mathbb{Q}_\ell)^G$. As both q and $c \circ q$ are finite maps and $\mathbb{Q}[G]$ for a finite group G is a semisimple $\mathbb{Q}$-algebra by the Maschke's Theorem, we acquire

$$\mathrm{R}c_*\mathbb{Q}_\ell \simeq \mathrm{R}c_*(q_*\mathbb{Q}_\ell)^G \simeq ((c \circ q)_*\mathbb{Q}_\ell)^G \simeq \mathbb{Q}_\ell$$

■

As the absolute Galois group of finite fields $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ acting on the ℓ-adic étale cohomology is a procyclic group that is topologically generated by the geometric Frobenius, the task of finding the eigenvalues of Frobenius map can be achieved through the trace formula where the cardinality of the fixed set of $\mathrm{Frob}_q : \mathfrak{X}(\overline{\mathbb{F}}_q) \to \mathfrak{X}(\overline{\mathbb{F}}_q)$ coincides with $\#_q(\mathfrak{X})$ the weighted $\mathbb{F}_q$–point count. We recall the *Grothendieck-Lefschetz trace formula* [Sun, Behrend] for Artin stacks of finite type over finite fields.

Theorem 2.10 (Theorem 1.1. of [Sun])**.** *Let $\mathfrak{X}$ be an Artin stack of finite type over $\mathbb{F}_q$. Let* Frob_q *be the geometric Frobenius on $\mathfrak{X}$. Let ℓ be a prime number different from the characteristic of $\mathbb{F}_q$, and let $\iota : \overline{\mathbb{Q}}_\ell \xrightarrow{\sim} \mathbb{C}$ be an isomorphism of fields. For an integer i, let $H^i_c(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)$ be the cohomology with compact support of the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\mathfrak{X}$ as in* [LO]*. Then the infinite sum regarded as a complex series via ι*

$$\sum_{i\in\mathbb{Z}}(-1)^i tr\big(\mathrm{Frob}_q^* : H^i_c(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell) \to H^i_c(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \overline{\mathbb{Q}}_\ell)\big) \tag{1}$$

is absolutely convergent to $\#_q(\mathfrak{X})$ over $\mathbb{F}_q$. And its limit is the number of $\mathbb{F}_q$–points on stacks that are counted with weights, where a point with its stabilizer group G contributes a weight $\frac{1}{|G|}$.

As $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ is a separated Deligne–Mumford stack of finite type over $\mathbb{F}_q$ (see Proposition 2.6), the corresponding (compactly supported) ℓ-adic étale cohomology for prime number ℓ invertible in $\mathbb{F}_q$ is finite dimensional as a $\mathbb{Q}_\ell$-algebra, making the trace formula hold in $\mathbb{Q}_\ell$-coefficients. Also by the smoothness of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ (see Proposition 2.6), the Poincaré duality $H^i_{\acute{e}t,c}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^{2dim(\mathfrak{X})-i}_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell(-dim(\mathfrak{X})))^\vee$ hold for the dual of ordinary étale cohomology as in [LO, 7.3.1. Theorem]. From now on, we identify the dual spaces with the ordinary étale cohomology as finite dimensional $\mathbb{Q}_\ell$-vector spaces. We denote by $\mathbb{Q}_\ell(-i)$ the $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-representation of rank 1 on which the *geometric Frobenius* acts by q^{-i} and the *arithmetic Frobenius* acts by q^i. Also, recall that the étale cohomology of a stack $\mathfrak{X}$ is *pure* if the absolute values of the eigenvalues of Frob_q on $H^i_{\acute{e}t}$ are all $q^{i/2}$ and if otherwise it is *mixed*. The group $H^i_{\acute{e}t}(\mathfrak{X}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ is of *Tate type* if the eigenvalues of Frob_q are all equal to integer powers of q.

We are now ready to prove the Theorem 1.2.

2.1. **Proof of Theorem 1.2.**

Proof. We first recall the étale cohomology of the smooth proper base $\mathcal{P}(a,b)$ the 1-dimensional weighted projective stack.

Proposition 2.11. *There are isomorphisms of Galois representations*

$$H^i_{\acute{e}t}(\mathcal{P}(a,b)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-1) & i=2 \\ \mathbb{Q}_\ell(0) & i=0 \\ 0 & else \end{cases}$$

Proof. This follows from [Kawasaki, Theorem 1.]. ∎

Next, we need the compactly supported étale cohomology of $\mathrm{Hom}^*_n(\mathbb{P}^1, \mathcal{P}(a,b)) \cong \mathrm{Poly}_1^{(an,bn)}$ which follows from [HP, FW].

Proposition 2.12. *There are isomorphisms of Galois representations*

$$H^i_{\acute{e}t,c}(\mathrm{Poly}_1^{(an,bn)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) \cong \begin{cases} \mathbb{Q}_\ell(-(an+bn)) & i=2(an+bn) \\ \mathbb{Q}_\ell(-(an+bn-1)) & i=2(an+bn)-1 \\ 0 & else \end{cases}$$

Proof. We can express $\mathrm{Poly}_1^{(an,bn)}$ as the quotient of a finite group $S_{an}\times S_{bn}$ acting on the complement of a hyperplane arrangement in $\mathbb{A}^{an+bn}$. It is well known by [Kim, BE] that the geometric Frobenius Frob_q^* acts by q^i on $H^i_{\acute{e}t}$ of the complement of a hyperplane arrangement. By transfer and Poincaré duality, we see that the geometric Frobenius acts by $q^{-(an+bn)+i}$ on $H^i_{\acute{e}t,c}(\mathrm{Poly}_1^{(an,bn)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$. Applying the *Grothendieck-Lefschetz trace formula* for compactly supported étale cohomology, we see that

$$\left|\mathrm{Poly}_1^{(an,bn)}(\mathbb{F}_q)\right| = \sum_{i=0}^{2(an+bn)} (-1)^i\cdot q^{-(an+bn)+i}\cdot\dim_{\mathbb{Q}_\ell}\left(H^i_{\acute{e}t,c}(\mathrm{Poly}_1^{(an,bn)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)\right)$$

and we use the exact point count of Proposition 2.8 to conclude the above. ∎

We now consider the ℓ-adic Leray spectral sequence as in [Behrend, Theorem 1.2.5] with respect to the evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b)) \to \mathcal{P}(a,b)$ to determine the ℓ-adic étale Betti numbers of $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b))$.

Proposition 2.13. *Evaluation morphism* $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b)) \to \mathcal{P}(a,b)$ *mapping an equivalence class of degree $n\in\mathbb{N}$ unbased morphism $f:\mathbb{P}^1\to\mathcal{P}(a,b)$ with the chosen basepoint $\infty\in\mathbb{P}^1$ to the basepoint of the target $f(\infty)\in\mathcal{P}(a,b)$ is a flat surjective morphism where $H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ the compactly supported ℓ-adic étale cohomology of the fiber at each geometric point $p\in\mathcal{P}(a,b)$ is isomorphic to $H^j_{\acute{e}t,c}(\mathrm{Poly}_1^{(an,bn)}{}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ the compactly supported ℓ-adic étale cohomology of the space $\mathrm{Poly}_1^{(an,bn)} \cong \mathrm{Hom}_n^*(\mathbb{P}^1,\mathcal{P}(a,b))$ of degree $n\in\mathbb{N}$ based morphisms.*

Proof. We have a $\mathbb{G}_m$-equivariant map $\widetilde{\mathrm{ev}}_\infty : T\to\mathbb{A}^2\setminus 0$, which induces $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b)) = [T/\mathbb{G}_m] \to \mathcal{P}(a,b) = [(\mathbb{A}^2\setminus 0)/\mathbb{G}_m]$. Note that for such maps and a closed subscheme $(x,y)\in\mathbb{A}^2\setminus 0$ on which $\mathbb{G}_m$ acts, $\mathrm{ev}_\infty^{-1}([x:y]) \cong \widetilde{\mathrm{ev}}_\infty^{-1}((x,y))/\mathbb{G}_m$. This implies in particular that the special fiber at $[1:0]$ or $[0:1]$ is the complement of the hyperplane arrangement by the same principle that the generic fiber such as over at $[1:1]$ is the complement of the hyperplane arrangement. The fiber at the generic stacky points $[x:y]\in\mathcal{P}(a,b)\setminus\{[0:1],[1:0]\}$ with $\mu_{gcd(a,b)}$ stabilizer can be accounted for by looking at $\mathrm{ev}_\infty^{-1}([1:1])$ where we have $\mathrm{ev}_\infty^{-1}([1:1]) \cong \mathrm{Poly}_1^{(an,bn)}$. This is because a pair of monic coprime polynomials (u,v) with $\deg(u,v) = (an,bn)$ determines, and is determined by, a morphism $f : \mathbb{P}^1 \to \mathcal{P}(a,b)$ such that $f^*\mathcal{O}_{\mathcal{P}(a,b)}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(n)$ and $f(\infty) = [1:1]$ via $f(z) := [u(z):v(z)]$ under the weighted homogeneous coordinate of the target stack $\mathcal{P}(a,b)$. By the $\mathbb{G}_m$ action, this implies that the compactly supported cohomology $H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ away from the special orbifold points $[0:1]$ or $[1:0]$ is isomorphic to $H^j_{\acute{e}t,c}({\mathrm{Poly}_1^{(an,bn)}}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$. For the compactly supported cohomology of the special fibers at $[0:1]$ or $[1:0]$, we have the fiber at $[0:1]$ with μ_b stabilizer that has the stratification $\mathrm{ev}_\infty^{-1}([0:1]) = \left(\coprod_{k=1}^{an} \mathbb{G}_m \times \mathrm{Poly}_1^{(an-k,bn)}\right)$ as it must map to $[0:1]$ by ev_∞. By applying the Grothendieck relation [Ekedahl, §1] on the stratification as in [HP], we have $[\mathbb{G}_m \times \mathrm{Poly}_1^{(an-1,bn)}] + [\mathbb{G}_m \times \mathrm{Poly}_1^{(an-2,bn)}] + \cdots + [\mathbb{G}_m \times \mathrm{Poly}_1^{(0,bn)}] = \mathbb{L}^{an+bn} - \mathbb{L}^{an+bn-1} = [\mathrm{Poly}_1^{(an,bn)}]$. Similarly for $[1:0]$ with μ_a stabilizer and the stratification of the fiber $\mathrm{ev}_\infty^{-1}([1:0]) = \left(\coprod_{k=1}^{bn} \mathbb{G}_m \times \mathrm{Poly}_1^{(an,bn-k)}\right)$ leading to $[\mathbb{G}_m \times \mathrm{Poly}_1^{(an,bn-1)}] + [\mathbb{G}_m \times \mathrm{Poly}_1^{(an,bn-2)}] + \cdots + [\mathbb{G}_m \times \mathrm{Poly}_1^{(an,0)}] = \mathbb{L}^{an+bn} - \mathbb{L}^{an+bn-1} = [\mathrm{Poly}_1^{(an,bn)}]$ which implies that both of the special fibers have the same $\mathbb{F}_q$–point counts as $\mathrm{Poly}_1^{(an,bn)}$. We apply the Proposition 2.12 to conclude that the compactly supported ℓ-adic étale cohomology of the special fibers at $[1:0]$ or $[0:1]$ are isomorphic to $H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) \cong H^j_{\acute{e}t,c}({\mathrm{Poly}_1^{(an,bn)}}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ as well. Flatness follows from the fibers having the constant dimension of $an+bn$. ■

Since the base $\mathcal{P}(a,b)$ is simply-connected (c.f. [Noohi, Example 9.2]), the stalk of the locally constant sheaves with compact support $(\mathrm{R}^j\,\mathrm{ev}_{\infty !}\,\mathbb{Q}_\ell)_p$ along a faithfully flat morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b)) \to \mathcal{P}(a,b)$ over a geometric point $p \in \mathcal{P}(a,b)$ can be identified with $H^j_{\acute{e}t,c}(\mathrm{ev}_\infty^{-1}(p)_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ by the base change for the cohomology with compact support as in [DIS, Section 6.3] (see also [Petersen, §2]). Working fiberwise, we have $\mathrm{R}^j\,\mathrm{ev}_{\infty !}\,\mathbb{Q}_\ell = H^j_{\acute{e}t,c}({\mathrm{ev}_\infty^{-1}}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell) = H^j_{\acute{e}t,c}({\mathrm{Poly}_1^{(an,bn)}}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$ by the Proposition 2.13 which implies that the ℓ-adic Leray spectral sequence has the following E_2 page through the Poincaré duality.

$$E_2^{i,j} = \begin{cases} H^i_{\acute{e}t}(\mathcal{P}(a,b)_{/\overline{\mathbb{F}}_q}; H^j_{\acute{e}t}({\mathrm{Poly}_1^{(an,bn)}}_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)) & \text{if } i,j \geq 0 \\ 0 & \text{else} \end{cases}$$

and the spectral sequence converges to $H^{i+j}_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)$.

Proposition 2.14. *ℓ-adic rational cohomology type of* $\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b))$ *is a 3-sphere with étale Betti numbers* $\dim_{\mathbb{Q}_\ell}\left(H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1,\mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q};\mathbb{Q}_\ell)\right)$

equal to one for $i = 0$, $i = 3$ and vanishes for all other i. By the Poincaré duality, we have $H^i_{\acute{e}t,c}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ one-dimensional for $i = 2(a+b)n+2$, $i = 2(a+b)n-1$ and vanishes for all other i.

Proof. Consider the coarse moduli map $c : \mathcal{P}(a,b) \to \mathbb{P}^1$ where c can be identified with $c([x : y]) = [x^{\mathrm{lcm}(a,b)/a} : y^{\mathrm{lcm}(a,b)/b}] \in \mathbb{P}^1$ for any $[x : y] \in \mathcal{P}(a,b) \cong [(\mathbb{A}^2_{x,y} \setminus 0)/\mathbb{G}_m]$. Since each coordinate function of $\mathbb{P}^1$ lifts to degree $\mathrm{lcm}(a,b)$ functions on $\mathcal{P}(a,b)$, we conclude that $c^*\mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_{\mathcal{P}(a,b)}(\mathrm{lcm}(a,b))$. This implies that $\deg(c \circ \varphi_f) = \mathrm{lcm}(a,b) \cdot \deg \varphi_f$ where $\deg \varphi_f := \deg \varphi_f^* \mathcal{O}_{\mathcal{P}(a,b)}(1) = n$. Then the ℓ-adic étale cohomology of the fine moduli stack $\mathcal{P}(a,b)$ and its coarse moduli space $\mathbb{P}^1$ are isomorphic through the induced map $c^* : H^i_{\acute{e}t}(\mathbb{P}^1{}_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \cong H^i_{\acute{e}t}(\mathcal{P}(a,b)_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ by Lemma 2.9. By the fuctoriality of the spectral sequence with regard to c^*, once we determine the transgression differential $d_2^{(2,0)}$ is an isomorphism for the case of $\mathrm{Hom}_{(\mathrm{lcm}(a,b)\cdot n)}(\mathbb{P}^1, \mathbb{P}^1)$ then we have the transgression differential $d_2^{(2,0)}$ is an isomorphism for the case of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ as well. By the comparison as in [FW, Theorem 1.2.(3)], we consider $H^i_{\mathrm{sing}}(\mathrm{Hom}_{(\mathrm{lcm}(a,b)\cdot n)}(\mathbb{P}^1, \mathbb{P}^1)(\mathbb{C}); \mathbb{C})$ where we have [KS, Proposition 2.2] showing that the transgression differential $d_2^{(2,0)}$ is an isomorphism for $\mathrm{Hom}_{(\mathrm{lcm}(a,b)\cdot n)}(\mathbb{P}^1, \mathbb{P}^1)$. As the transgression differential $d_2^{(2,0)}$ is also an isomorphism for $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$, this implies that the ℓ-adic étale Betti numbers $\dim_{\mathbb{Q}_\ell} \left(H^i_{\acute{e}t}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell) \right)$ are equal to one for $i = 0$, $i = 3$ and vanishes for all other i. ∎

As $H^i_{\acute{e}t,c}(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))_{/\overline{\mathbb{F}}_q}; \mathbb{Q}_\ell)$ are all one dimensional, the trace of Frob_q on each of these $\mathbb{Q}_\ell$-vector spaces is just the corresponding eigenvalue λ_i of Frob_q.

When $i = 2(an+bn)+2$, the connectedness of $\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))$ together with the Poincaré duality implies that $\lambda_{2(an+bn)+2} = q^{(a+b)n+1}$

Plugging all of the above into the *Grothendieck-Lefschetz trace formula* (2.10) :

$$\#_q(\mathrm{Hom}_n(\mathbb{P}^1, \mathcal{P}(a,b))) = q^{(a+b)n+1} - q^{(a+b)n-1} = \lambda_{2(an+bn)+2} - \lambda_{2(an+bn)-1}$$

$$= q^{(a+b)n+1} - \lambda_{2(an+bn)-1}$$

which implies that $\lambda_{2(an+bn)-1} = q^{(a+b)n-1}$ as claimed.

This finishes the proof of Theorem 1.2.

∎

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Center for Geometry and Physics, Institute for Basic Science (IBS), Pohang 37673, Korea

E-mail address: `junepark@ibs.re.kr`