Elliptic curves over $k(t)$, $\mathbb{F}_q(t)$, $\mathbb{C}(t)$: Height moduli, exact counts, rank jumps

Abstract. The notion of height does more than order the infinite set of elliptic curves: it turns their totality into a geometric space. After a brief motivation over $\mathbb{Q}$, I will explain this principle over three rational function fields.

Over $k(t)$, an elliptic curve can be viewed as an elliptic surface over $\mathbb{P}^1_k$, and its height is the degree of its classifying map to $\overline{\mathcal{M}}_{1,1}$. The resulting height moduli stack is of finite type and parametrizes elliptic curves of fixed Faltings height n, with a natural correspondence between its strata and their Kodaira fibre configurations.

Over $\mathbb{F}_q(t)$, motivic identities in the Grothendieck ring of stacks specialize to exact counts of elliptic curves of bounded height in every characteristic, including $p = 2$ and $p = 3$. Every lower-order term can be traced to a special stratum, automorphism locus, or minimality defect.

Over $\mathbb{C}(t)$, fixing a Kodaira fibre stratum and applying the Shioda-Tate formula identifies Mordell-Weil rank jumps with additional Hodge classes. Special cycle modularity organizes their possible height pairings, while a separate period map argument produces analytically dense rank jumps throughout the expected transverse range of $1 \leq r \leq \left\lfloor \frac{10n-2}{n-1} \right\rfloor$.