100% of elliptic curves with a marked point have positive rank

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Abstract

As a consequence of their work on average Selmer ranks of elliptic curves with marked points [BH22, §10], Bhargava and Ho prove that 100% of elliptic curves over $\mathbb{Q}$ with an additional marked point have positive rank. In this note we provide an alternate proof which extends the result to arbitrary global fields.

Let $\mathcal{M}_{1,2}$ denote the moduli space of genus one curves with two marked points. Let K be a global field of characteristic not equal to 2 or 3. Over K the space $\mathcal{M}_{1,2}$ can be identified with an open substack of the weighted projective stack $\mathcal{P}(2,3,4)$ (see, e.g., [Inc22, §2]). We define a height function on $\mathcal{M}_{1,2}$ by defining a height on $\mathcal{P}(2,3,4)$.

Let $\mathrm{Val}(K)$ denote the set of places of K, and let $\mathrm{Val}_0(K)$ (resp. $\mathrm{Val}_\infty(K)$) denote the set of finite places (resp. infinite places) of K. For each finite place $v \in \mathrm{Val}_0(K)$ let π_v be a uniformizer. For $x = (x_0, x_1, x_2) \in K^3 - \{(0,0,0)\}$ define

$$|x|_{(2,3,4),v} \stackrel{\text{def}}{=} \begin{cases} \max\left\{ |\pi_v|_v^{\lfloor v(x_0)/2 \rfloor}, |\pi_v|_v^{\lfloor v(x_1)/3 \rfloor}, |\pi_v|_v^{\lfloor v(x_2)/4 \rfloor} \right\} & \text{if } v \in \mathrm{Val}_0(K), \\ \max\left\{ |x_0|_v^{1/2}, |x_1|_v^{1/3}, |x_2|_v^{1/4} \right\} & \text{if } v \in \mathrm{Val}_\infty(K). \end{cases} \tag{1}$$

Then the *height* of a point $x = [x_0 : x_1 : x_2] \in \mathcal{P}(2,3,4)(K)$ is defined to be

$$\mathrm{Ht}_{(2,3,4)}(x) \stackrel{\text{def}}{=} \prod_{v \in \mathrm{Val}(K)} |(x_0, x_1, x_2)|_{\mathbf{w},v}. \tag{2}$$

Theorem 1. *Let K be a global field of characteristic not equal to 2 or 3. When ordered by height, 100% of elliptic curves defined over K with an additional marked point have positive rank.*

Proof. It follows from [Phi24, Theorem 4.1.1] and [Dar21, Theorem 8.3.2.2] (see also [Phi25, Theorem 4.05]) that for any open substack $X \subseteq \mathcal{P}(2,3,4)$ we have the asymptotic

$$\#\{x \in X(K) : \mathrm{Ht}_{(2,3,4)}(x) \leq B\} \sim \#\{x \in \mathcal{P}(2,3,4)(K) : \mathrm{Ht}_{(2,3,4)}(x) \leq B\}. \tag{3}$$

Let $\mathcal{Y}_1(N)$ be the modular curve parameterizing pairs (E, P), where E is an elliptic curve and P is a point of order N. View $\mathcal{Y}_1(N)$ as a substack of $\mathcal{M}_{1,2}$. Let $\mathcal{Y}_{tors}$ denote the union of all modular curves $\mathcal{Y}_1(N)$ with $\mathcal{Y}_1(N)(K) \neq \emptyset$. By uniform bounds for torsion of elliptic curves [Lev68, Mer96], $\mathcal{Y}_{tors}$ will be a finite union, and thus a closed substack of $\mathcal{M}_{1,2}$. Applying equation (3) with $X = \mathcal{M}_{1,2}$ and $X = \mathcal{M}_{1,2} - \mathcal{Y}_{tors}$ proves the theorem. $\square$

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