HEIGHT MODULI OF ELLIPTIC SURFACES: MOTIVIC HEIGHT ZETA RATIONALITY AND KUDLA–MILLSON MODULARITY OF MORDELL–WEIL RANK JUMPS

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ABSTRACT. Let k be a perfect field with $\mathrm{char}(k) \neq 2, 3$, set $K = k(t)$, and let $\mathcal{W}_n^{\min}$ be the moduli stack of minimal elliptic curves over K of Faltings height n, constructed via the height–moduli framework of [BPS22] applied to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$. The Shioda–Tate formula $\rho(S) = T(S) + \mathrm{rk}(E/K)$ decomposes the Picard rank of the associated elliptic surface into the trivial lattice rank, which is local (determined by Kodaira fiber types), and the Mordell–Weil rank, which is global. The motivic height zeta function weighted by the trivial lattice rank is rational in $s = t^{1/12}$ in the dimensionally completed Grothendieck ring, via a combination of exact Euler products on the isotrivial loci $j \equiv 0, 1728$ and a motivic discriminant stabilization adapting Vakil–Wood [VW15] to $\Delta = 4a_4^3 + 27a_6^2$; over $k = \mathbb{C}$, this yields bidegree-wise Hodge number stabilization. The Kudla–Millson theta correspondence [KM90] shows that the distribution of new Mordell–Weil sections by canonical height is governed by a modular form of weight $6n-2$ for $\mathrm{SL}_2(\mathbb{Z})$. Combining Shepherd-Barron's diagonalization of the Gauss–Manin connection [SB23] with Kodaira–Spencer transversality, we establish unconditionally that at every Faltings height $n \geq 3$ and for every $1 \leq r \leq \lfloor (10n-2)/(n-1) \rfloor$, there exist infinitely many stable elliptic surfaces with Mordell–Weil rank $\mathrm{rk}(E/K) \geq r$, and that infinitely many canonical heights $\hat{h}(P) = d$ are realized by Mordell–Weil sections.

1. INTRODUCTION

Let k be a perfect field with $\mathrm{char}(k) \neq 2, 3$, and set $K := k(t)$. An elliptic curve E/K determines a relatively minimal elliptic surface with section

$$f : S \longrightarrow \mathbb{P}^1_k$$

unique up to isomorphism (see [Mir89, SS10] for background on elliptic surfaces).

The arithmetic of E/K is reflected in the geometry of S, and a basic organizing principle is the Shioda–Tate formula [Shi90]

$$\rho(S) = T(S) + \mathrm{rk}(E/K), \tag{1}$$

where $\rho(S) = \mathrm{rk}\,\mathrm{NS}(S_{\bar{k}})$ is the *geometric Picard rank*, $T(S)$ is the *rank of the geometric trivial lattice* generated by the zero section, a fiber class, and the components of reducible fibers not meeting the zero section, and $\mathrm{rk}(E/K)$ is the *Mordell–Weil rank*. For the relatively minimal elliptic surfaces $f : S \to \mathbb{P}^1_k$ with section considered in this paper, we have $q(S) = 0$ and $p_g(S) = n-1$, hence the standard bounds

$$2 \leq \rho(S) \leq 10n, \qquad 2 \leq T(S) \leq 10n, \qquad 0 \leq \mathrm{rk}(E/K) \leq 10n-2, \tag{2}$$

where $\rho(S) \leq 10n = h^{1,1}(S)$ is the Lefschetz bound over $k = \mathbb{C}$ (or in general Igusa's inequality $\rho(S) \leq b_2(S) = 12n-2$).

In [BPS22], Bejleri–Park–Satriano construct height-moduli stacks of rational points on proper polarized cyclotomic stacks. In the fundamental modular curve example

$$\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6), \qquad \lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1),$$

a minimal elliptic curve over K can be viewed as a rational point of λ–height n on $\overline{\mathcal{M}}_{1,1}$ over K. This yields a separated Deligne–Mumford stack of finite type

$$\mathcal{W}_n^{\min} \coloneqq \mathcal{W}_{n,\mathbb{P}^1_k}^{\min}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)$$

parametrizing minimal elliptic curves over K of discriminant degree $12n$. Here a K-rational point of $\overline{\mathcal{M}}_{1,1}$ of λ-height n means the stacky height n with respect to the Hodge line bundle λ, in the sense of [ESZB23, Def. 2.11]. Under the identification $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ one has $\lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1)$, and this height agrees with the Faltings height of the corresponding elliptic curve by [BPS22, Cor. 7.6].

Guided by (1), we introduce the following motivic generating series (see [Eke25] for background on the Grothendieck ring of stacks) refining the height generating series in [BPS22, §8] by weighting each height stratum with the *lattice ranks* of the associated relatively minimal elliptic surface.

Definition 1.1. Let k be a perfect field of characteristic $\neq 2,3$, and consider the height–moduli stack

$$\mathcal{W}_n^{\min} = \mathcal{W}_{n,\mathbb{P}^1_k}^{\min}\big(\mathcal{P}(4,6),\mathcal{O}(1)\big)$$

parametrizing minimal elliptic curves over $K = k(t)$ of discriminant height $12n$. The *formal trivariate height zeta function* is

$$\mathcal{Z}(u,v;t) \coloneqq \sum_{n\geq 0}\left(\sum_{[E]\in\mathcal{W}_n^{\min}} u^{T(S)}\, v^{\mathrm{rk}(E/K)}\right) t^n$$

where for each $[E] \in \mathcal{W}_n^{\min}$ we write $S \to \mathbb{P}^1_k$ for the associated relatively minimal elliptic surface $f : S \to \mathbb{P}^1_k$ with section, and:

- $T(S)$ is the rank of the trivial lattice of S;
- $\mathrm{rk}(E/K)$ is the Mordell–Weil rank.

The following specializations are the associated *bivariate height zeta functions*:

$$Z_{\mathrm{Triv}}(u;t) \coloneqq \mathcal{Z}(u,1;t), \tag{3}$$

$$Z_{\mathrm{MW}}(v;t) \coloneqq \mathcal{Z}(1,v;t), \tag{4}$$

$$Z_{\mathrm{NS}}(w;t) \coloneqq \mathcal{Z}(w,w;t). \tag{5}$$

The specialization $Z_{\mathrm{Triv}}(u;t) = \mathcal{Z}(u,1;t)$ is rigorously defined in $K_0(\mathrm{Stck}_k)[u][\![t]\!]$, since $T(S)$ is constant on each Kodaira stratum. The specializations $Z_{\mathrm{MW}}(v;t)$ and $Z_{\mathrm{NS}}(w;t)$ are well-defined as weighted point counting series over $\mathbb{F}_q$ but not motivically in general: for a given Mordell–Weil rank r, there are infinitely many Néron–Severi lattice types realizing that rank, each contributing a separate Noether–Lefschetz stratum [CDK95], so the rank-r locus is a countable union that is not constructible. We therefore work motivically in u throughout and treat the v- and w-gradings as weighted point counting refinements.

Setting $u = v = 1$ forgets the lattice rank grading and specializes to the *univariate motivic height zeta function* $Z_{\tilde{\lambda}}(t) = \mathcal{Z}(1,1;t) \in K_0(\mathrm{Stck}_k)[\![t]\!]$ and likewise to

its inertial refinement $\mathcal{I}Z_{\tilde{\lambda}}(t)$ which encodes the totality of rational points on $\overline{\mathcal{M}}_{1,1}$ over $K = k(t)$. [BPS22, Thm. 8.9] shows that both series are in fact rational in t, i.e. lie in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}](t)$, and gives explicit formulas.

The assumption $\mathrm{char}(k) \neq 2, 3$ is used throughout in two essential ways: first, it ensures the existence of the short Weierstrass form $y^2 = x^3 + a_4x + a_6$ (equivalently, the isomorphism $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$ over $\mathbb{Z}\left[\frac{1}{6}\right]$); second, it guarantees that the Kodaira–Néron fiber classification [Kod63, Né64] and the Tate correspondence (i.e. *Tate's algorithm* [Tat75] *via twisted maps* [BPS22, Thm. 7.12]) apply in their standard form.

We first focus on $Z_{\mathrm{Triv}}(u;t)$. The key point is that the trivial lattice is governed by *local bad reduction*: its rank is determined by the geometric Kodaira fiber configuration of $\pi_{\bar{k}} \colon S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. Writing $\mathrm{Triv}(S) \subset \mathrm{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) := \mathrm{rk}(\mathrm{Triv}(S))$, we have the following explicit formula.

Lemma 1.2. *Let $\pi \colon S \to \mathbb{P}^1_k$ be a relatively minimal elliptic surface with section, and let $\mathfrak{f}$ be the multiset of singular fibers of $\pi_{\bar{k}} \colon S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$. If m_v denotes the number of irreducible components of the fiber at v, then*

$$T(S) \;=\; 2 + \sum_{v \in \mathfrak{f}} (m_v - 1).$$

Definition 1.3. Fix $n \geq 1$. For a geometric fiber configuration $\mathfrak{f}$, the *Kodaira stratum* $\mathcal{W}_n^{\min,(\mathfrak{f})} \subset \mathcal{W}_n^{\min}$ is the locus parametrizing those $[E] \in \mathcal{W}_n^{\min}$ whose associated surface $S_{\bar{k}} \to \mathbb{P}^1_{\bar{k}}$ has singular fiber configuration $\mathfrak{f}$ (cf. [BPS22, Thms. 5.1 and 7.12]).

Definition 1.4. Fix $n \geq 0$. By Proposition 2.1, $\mathcal{W}_n^{\min}$ admits a *constructible stratification* by Kodaira data, and $T(S)$ is constant on each stratum. For $n \geq 1$ and each T with $2 \leq T \leq 10n$, let

$$\mathcal{W}_n^{\min}(T) \subset \mathcal{W}_n^{\min}$$

be the finite union of those Kodaira strata on which $T(S) = T$ (hence a finite union of locally closed substacks). For $n = 0$, set $\mathcal{W}_0^{\min} := \mathcal{W}_0^{\min}(2)$.

The trivial–lattice–rank–weighted motivic height zeta function is

$$Z_{\mathrm{Triv}}(u;t) := \sum_{n \geq 0} \sum_{T \geq 2} u^T \left\{\mathcal{W}_n^{\min}(T)\right\} t^n \;\in\; K_0(\mathrm{Stck}_k)[u][\![t]\!].$$

We prove that $Z_{\mathrm{Triv}}(u;t)$ is approximately rational in $s = t^{1/12}$ in the dimensionally completed Grothendieck ring, reflecting the local nature of the trivial lattice.

Theorem 1.5. *Let k be a perfect field with $\mathrm{char}(k) \neq 2, 3$, put $s = t^{1/12}$, and let $\widehat{\mathcal{M}}_{\mathbb{L}}$ denote the completion of $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$ with respect to the dimensional filtration* [Kon95, VW15]. *Then for each $m \geq 2$ and each $N \geq 0$, there exists a rational function $R_{m,N}(s) \in \widehat{\mathcal{M}}_{\mathbb{L}}(s)$ such that for all n sufficiently large (depending on m and N), the normalized error*

$$\left([u^m] Z_{\mathrm{Triv}}(u;t)\Big|_{t^n} - R_{m,N}(s)\Big|_{s^{12n}}\right) \cdot \mathbb{L}^{-10n}$$

has dimension $\leq -(N+1)$ in $\widehat{\mathcal{M}}_{\mathbb{L}}$. Equivalently,

$$[u^m] Z_{\mathrm{Triv}}(u;t) \;\in\; \overline{\widehat{\mathcal{M}}_{\mathbb{L}}(s)} \;\subset\; \widehat{\mathcal{M}}_{\mathbb{L}}[\![s]\!]$$

for each m, where the closure is taken with respect to the coefficient-wise dimensional filtration topology.

The approximating rational functions arise from a three-locus partition of the moduli ($j \equiv 0$, $j \equiv 1728$, and the complement):

(i) *On the isotrivial loci $j \equiv 0$ and $j \equiv 1728$, the discriminant is a pure power of a single Weierstrass coefficient, all local conditions are linear, and the resulting Euler products are exact at all heights. These are* strictly *rational: they lie in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}](s)$ itself, without completion or approximation.*

(ii) *On the complement where $j \not\equiv 0, 1728$, the additive fiber types are handled by linear independence of vanishing conditions on the Weierstrass coefficients, yielding an Euler product that is exact above an explicit height threshold.*

(iii) *The multiplicative residual (root multiplicities of $\Delta = 4a_4^3 + 27a_6^2$ at coprime points) is handled by a motivic discriminant stabilization adapting the method of Vakil–Wood* [VW15]*: a truncated inclusion–exclusion at depth N produces the rational approximation $R_{m,N}$, with error vanishing in the dimensional filtration as $N \to \infty$.*

Setting $u = 1$ recovers the exact Euler product of [BPS22, Thm. 8.9]*: the approximation becomes exact, and strict rationality holds.*

Over $k = \mathbb{C}$, the Hodge–Deligne motivic measure $\mathrm{HD}\colon K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}] \to \mathbb{Z}[x, y, (xy)^{-1}]$, $\mathbb{L} \mapsto xy$, extracts compactly-supported virtual Hodge numbers $e_c^{p,q}$ from motivic classes. The weight bound in mixed Hodge theory [Eke25] converts the dimensional error bounds of Theorem 1.5 into exact stabilization at each bidegree.

Corollary 1.6. *Let $k = \mathbb{C}$, $K = \mathbb{C}(t)$, and $s = t^{1/12}$. Fix $m \geq 2$. For each bidegree $(p_0, q_0) \in \mathbb{Z}^2$, the virtual Hodge number*

$$e_c^{p_0,q_0}\Big(\big\{\mathcal{W}_n^{\min}(m)\big\} \cdot \mathbb{L}^{-10n}\Big)$$

is independent of n for all n sufficiently large, where the threshold depends on m and (p_0, q_0). In particular, the generating function $\sum_{n\geq 0} e_c^{p_0,q_0}(\cdots)s^{12n}$ is rational in s.

In sharp contrast with the local nature of the trivial lattice, we establish transcendence of the distribution of new algebraic classes by canonical height at every Faltings height $n \geq 2$, using the Kudla–Millson theta correspondence.

Theorem 1.7. *For each $n \geq 2$, let $\Lambda_n = \langle F, O\rangle^{\perp} \subset H^2(S, \mathbb{Z})$ be the orthogonal complement of the sublattice spanned by the fiber class and zero section, an even unimodular lattice of signature $(2n{-}2,\ 10n{-}2)$ and rank $12n{-}4$. Let $\mathcal{D}_n$ be the associated period domain.*

(1) *The generating series of Noether–Lefschetz classes*

$$c_{\mathrm{top}}(\mathcal{F}^{\vee}) \;+\; \sum_{d\geq 1} [\mathcal{H}_d]\, q^d \;\in\; H^{2(n-1)}(O(\Lambda_n, \mathbb{Z})\backslash \mathcal{D}_n,\, \mathbb{Q})[\![q]\!]$$

is a holomorphic modular form of weight $6n - 2$ for $\mathrm{SL}_2(\mathbb{Z})$, valued in cohomology $H^{2(n-1)}(O(\Lambda_n, \mathbb{Z})\backslash \mathcal{D}_n, \mathbb{Q})$. The Fourier coefficient $[\mathcal{H}_d]$ is the cycle class of the Heegner locus parametrizing Hodge structures where a lattice vector $\gamma \in \Lambda_n$ with $\gamma^2 = -2d$ becomes algebraic.

(2) *On any fixed Kodaira stratum* $\mathcal{W}_n^{\min,(\mathfrak{f})}$*, the root lattice* $R(\mathfrak{f}) \subset \Lambda_n$ *is constant and algebraic throughout. A class* $\gamma \in \Lambda_n$ *with* $\gamma^2 = -2d$ *that becomes newly algebraic on the stratum cannot lie in* $R(\mathfrak{f})$*, hence forces* $\mathrm{rk}(E/K) \geq 1$ *by the Shioda–Tate exact sequence. The corresponding section has canonical height* $\hat{h}(P) \leq d$*, with equality when all singular fibers are irreducible (types* I_1 *and* II*). The genus-*r *Siegel theta series restricted to* $R(\mathfrak{f})^\perp \cap \Lambda_n$ *detects* $\mathrm{rk}(E/K) \geq r$ *for each* $1 \leq r \leq 10n - T(\mathfrak{f})$*.*

(3) *For any test class* $\alpha \in H_{2(n-1)}(O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n, \mathbb{Q})$ *for which* $\alpha \cap [\mathcal{H}_d] \neq 0$ *for some* d*, the scalar-valued series* $\varphi_{n,\alpha}(q) = \alpha \cap c_{\mathrm{top}}(\mathcal{F}^\vee) + \sum_{d\geq 1}(\alpha \cap [\mathcal{H}_d])\, q^d \in M_{6n-2}(\mathrm{SL}_2(\mathbb{Z}))$ *is transcendental over* $\mathbb{C}(q)$*. Such* α *exist for all* $n \geq 2$*.*

Theorem 1.7 makes precise the structural distinction between the two summands in the Shioda–Tate formula. The trivial lattice rank $T(S)$ is determined by the Kodaira fiber configuration, which depends only on local reduction data at finitely many points of $\mathbb{P}^1$; this *locality* drives the Euler product structure underlying the approximate rationality of $Z_{\mathrm{Triv}}(u;t)$, with the approximation arising from the nonlinearity of the discriminant $\Delta = 4a_4^3 + 27a_6^2$. By contrast, a new Mordell–Weil section is a genuinely *global* object: it is a class in Λ_n outside the root lattice that becomes newly algebraic on a fixed Kodaira stratum, and the locus where this happens is a Heegner cycle in the period domain, governed by the theta correspondence. The generating series of these loci, indexed by the canonical height d of the new section, assembles into a modular form whose transcendence over $\mathbb{C}(q)$ reflects the genuinely analytic, non-constructible nature of Noether–Lefschetz loci, in sharp contrast to the constructible local conditions governing the trivial lattice.

The modularity of Noether–Lefschetz generating series for elliptic surfaces, in all Siegel genera, was established by Greer [Gre20, Thm. 37] by pulling back the Kudla–Millson theorem [KM90] to the period domain. Garcia [Gar18] gives an alternative proof using Quillen superconnections, working intrinsically on $\mathcal{D}_n$. The per-Kodaira-stratum argument that newly algebraic classes on a fixed Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$ have nonzero Mordell–Weil projection uses the constructible stratification from [BPS22, Thms. 5.1 and 7.12]. For elliptic K3 surfaces with $n = 2$ we have $\dim S_{10}(\mathrm{SL}_2(\mathbb{Z})) = 0$, so the modular form is purely Eisenstein: $\varphi_{2,\alpha} \propto E_{10}$ and the Noether–Lefschetz numbers are proportional to $\sigma_9(d)$, as computed by Maulik–Pandharipande [MP13] in the K3 setting. At $n = 3$ the cusp form $\Delta E_4 \in S_{16}(\mathrm{SL}_2(\mathbb{Z}))$ contributes nontrivially (Remark 3.6).

Theorem 1.7 is a statement on the period domain $O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n$. For $n \geq 3$ the period map $\Phi\colon \mathcal{W}_n^{\mathrm{st}} \to O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n$ from the *stable stratum* $\mathcal{W}_n^{\mathrm{st}} \subset \mathcal{W}_n^{\min}$ (the open substack parametrizing elliptic surfaces whose singular fibers are all of type I_1) is far from surjective (the moduli has dimension $10n-2$ while the period domain has dimension $(n-1)(10n-2) + \frac{1}{2}(n-1)(n-2)$), and it is not *a priori* clear that the Heegner loci $\mathcal{H}_d$ are met by the period image. We show that the infinitesimal period map is transverse to every Heegner locus, and that infinitely many canonical heights are geometrically realized by sections on stable elliptic surfaces.

Theorem 1.8. *Let* $k = \mathbb{C}$*,* $K = \mathbb{C}(t)$*, and* $n \geq 3$*.*

(1) *For every primitive $\gamma \in \Lambda_n$ with $\gamma^2 < 0$, the Kodaira–Spencer map of the universal Weierstrass family over $\mathcal{W}_n^{\mathrm{st}}$ is transverse to $\gamma^\perp$. In particular, $\Phi^{-1}(\mathcal{H}_d)$ is either empty or has codimension exactly $n-1$ in $\mathcal{W}_n^{\mathrm{st}}$ for every $d \geq 1$.*
(2) *For infinitely many $d \geq 1$, there exists a stable elliptic surface $\pi\colon S \to \mathbb{P}^1$ at Faltings height n with $\mathrm{rk}(E_S/K) \geq 1$ and a section of canonical height $\hat{h}(P) = d$.*
(3) *For every integer $1 \leq r \leq \lfloor (10n-2)/(n-1) \rfloor$, there exist infinitely many stable elliptic surfaces at Faltings height n with Mordell–Weil rank $\mathrm{rk}(E_S/K) \geq r$. The bound equals 14 for $n = 3$ and stabilizes at 10 for $n \geq 10$.*

The transversality uses the orthogonal basis of $H^{1,1}_{\mathrm{prim}}$ and the diagonalization of the Gauss–Manin connection established by Shepherd-Barron [SB23]; the realization combines this with the algebraicity of Noether–Lefschetz loci [CDK95] and the per-Kodaira-stratum Shioda–Tate argument of Theorem 1.7(2).

Remark 1.9. The bound $r \leq \lfloor (10n-2)/(n-1) \rfloor$ is the ratio of $\dim \mathcal{W}_n^{\mathrm{st}} = 10n-2$ to the codimension $h^{2,0}(S) = n-1$ of each Heegner locus in $\mathcal{D}_n$: each algebraic class imposes $n-1$ independent conditions on $\mathcal{W}_n^{\mathrm{st}}$ to become Hodge, and $r(n-1) \leq \dim \mathcal{W}_n^{\mathrm{st}}$ is the room. The ratio is 14 at $n = 3$ and stabilizes to 10 as $n \to \infty$. Any generic local deformation approach is bounded by this ratio. The known rank records exceed it via Delsarte constructions with large cyclic symmetry — Shioda's isotrivial $r = 68$ at $n = 60$ [Shi92] and the Stiller–Shioda non-isotrivial $r = 56$ at $n = 210$ [Shi86, Sti87] — which exploit special automorphisms unavailable to generic deformations. Whether Mordell–Weil rank is unbounded over $\mathbb{C}(t)$ as $n \to \infty$ remains open [SS19, Problem 13.1, Remark 13.29]. The analogous question over $\mathbb{F}_q(t)$ has a positive answer by Tate–Shafarevich [Tv67] and Ulmer [Ulm02], via Frobenius-induced algebraic cycles unavailable in characteristic zero.

1.1. **Proof methods for Theorem 1.5.** The proof combines a motivic local-to-global factorization in the style of Kapranov [Kap00, CLL16] with the twisted-map stratification of the height–moduli $\mathcal{W}_n^{\min}$ from [BPS22] and the evaluation morphisms of [BM96]. Throughout we use the Bejleri–Park–Satriano correspondence [BPS22, Thm. 3.3] between rational points, minimal weighted linear series, and twisted morphisms; in particular, local reduction conditions are encoded by representable twisted morphisms to $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$, yielding a moduli-theoretic Tate correspondence compatible with the minimal model program. Unordered collections of local factors supported at distinct points of $\mathbb{P}^1$ are governed by symmetric powers $\mathrm{Sym}^N(\mathbb{P}^1)$. We reorganize these symmetric-power contributions using the power structure on the Grothendieck ring $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$, and state the resulting identity in Lemma 2.7.

The additive local factors impose linear vanishing conditions on the Weierstrass coefficients (a_4, a_6), and the sparsity argument of Lemma 2.7 shows these are independent at large heights. On the isotrivial loci $j \equiv 0$ and $j \equiv 1728$—the same loci that produce the lower order main terms in [BPS22, Thm. 9.7]—the discriminant reduces to a pure power of a single Weierstrass coefficient, and all local conditions become linear. These loci admit their own exact Euler products (Proposition 2.8), with no correction at any height.

The multiplicative sector (cusp families I_k and I_k^*) presents a fundamental obstruction: the discriminant $\Delta = 4a_4^3 + 27a_6^2$ is a *nonlinear* function of the Weierstrass data, and the source space $H^0(\mathcal{O}(4n)) \oplus H^0(\mathcal{O}(6n))$ (dimension $10n+2$) is too small to independently prescribe discriminant root multiplicities at all singular points (the total discriminant degree is $12n$). The motivic transversality of Lemma 2.6 handles the regime where the multiplicative sector carries at most $\frac{1}{3}$ of the discriminant degree, but the generic semistable elliptic surface lies outside this range. To resolve this, we adapt the motivic stabilization method of Vakil–Wood [VW15]: for each fixed trivial lattice rank m, there are finitely many "visible" fiber configurations (those contributing positively to m), and the residual simple roots of Δ are handled by a truncated inclusion–exclusion that converges in the dimensional completion $\widehat{\mathcal{M}}_{\mathbb{L}}$ (Proposition 2.11). At $u = 1$ the trivial lattice grading is forgotten, the visible/invisible distinction collapses, and both the small-height additive corrections and the discriminant stabilization errors vanish identically, recovering the exact Euler product of [BPS22, Thm. 8.9].

2. Approximate rationality of the trivial lattice specialization

Throughout, let k be a perfect field with $\mathrm{char}(k) \neq 2, 3$, set $K = k(t)$, and let $\pi\colon S \to \mathbb{P}^1_k$ be the relatively minimal elliptic surface with section associated to E/K. Write $\mathrm{Triv}(S) \subset \mathrm{NS}(S_{\bar{k}})$ for the geometric trivial lattice and $T(S) = \mathrm{rk}\,\mathrm{Triv}(S)$.

Proposition 2.1. *Fix $n \geq 1$. The discriminant degree constraint $\sum_v e(F_v) = 12n$ implies that only finitely many geometric fiber configurations $\mathfrak{f}$ occur among surfaces parametrized by $\mathcal{W}_n^{\min}$. Consequently,*

$$\mathcal{W}_n^{\min} = \bigsqcup_{\mathfrak{f}} \mathcal{W}_n^{\min,(\mathfrak{f})}$$

is a constructible stratification. Moreover, the trivial lattice rank $T(S)$ is constant on each stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$.

Proof. The Kodaira–Néron classification gives $e(\mathrm{I}_k) = k$, $e(\mathrm{I}_k^*) = k+6$, and $e(F_v) \in \{2, 3, 4, 6, 8, 9, 10\}$ for the remaining types. Since $\sum_v e(F_v) = 12n$, only finitely many multisets of Kodaira symbols can occur, yielding the finite stratification by [BPS22, Thms. 5.1 and 7.12]. Constancy of $T(S)$ on each stratum follows from Lemma 1.2. ∎

2.1. **A multivariate height series.** We briefly recall the local indexing used in the twisted-maps description of height–moduli. By [BPS22, Thm. 5.1] the height-n moduli stack $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ on a proper polarized cyclotomic stack $\mathcal{X}$ with polarizing line bundle $\mathcal{L}$ admits a finite stratification by locally closed substacks indexed by admissible local conditions and degrees: there is a finite disjoint union of morphisms

$$\bigsqcup_{\Gamma,d} \mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})/S_\Gamma \longrightarrow \mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L}),$$

where $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X}, \mathcal{L})$ is the moduli stack of representable twisted morphisms of stable height d to $(\mathcal{X}, \mathcal{L})$ with local twisting conditions

$$\Gamma = \big(\{r_1, a_1\}, \ldots, \{r_s, a_s\}\big),$$

recording the stabilizer orders r_i and the corresponding characters a_i at the stacky marked points of the source root stack. The indices (Γ, d) range over those satisfying the height decomposition formula

$$n \;=\; d \;+\; \sum_{i=1}^{s} \frac{a_i}{r_i}.$$

Here $S_\Gamma \subset S_s$ is the subgroup permuting stacky marked points of the same local type.

Definition 2.2. For the Euler-product argument it is useful to distinguish *local factor types* from *evaluation labels*. Let $\mathcal{I}\overline{\mathcal{M}}_{1,1} \simeq \mathcal{IP}(4,6)$ be the inertia stack of $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$ over $\mathbb{Z}\left[\frac{1}{6}\right]$ with $\mathcal{L} = \mathcal{O}(1)$ the Hodge line bundle (see [HP23, §2] for background on inertia stacks).

(1) Local factor types. Let J denote the finite set of local factor types occurring in the Tate algorithm stratification via twisted maps (see [BPS22, §7]); concretely one may take

$$J = \Big\{\mathrm{II},\ \mathrm{III},\ \mathrm{IV},\ \mathrm{II}^*,\ \mathrm{III}^*,\ \mathrm{IV}^*,\ \mathrm{I}_0^*(j \neq 0, 1728),\ \mathrm{I}_0^*(j=0),\ \mathrm{I}_0^*(j=1728),\ \mathrm{I}_\bullet,\ \mathrm{I}_\bullet^*\Big\},$$

where $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$ are the two cusp *shapes* over $j = \infty$.

(2) Evaluation labels. Let $\mathcal{A}$ denote the set of evaluation labels used to index evaluation conditions, i.e. the inertia components in which the evaluation maps land. Away from the cusp $j = \infty$, the inertia label determines the Kodaira symbol, so the non-cusp labels form a finite set

$$\mathcal{A}_{\mathrm{nc}} = \Big\{\mathrm{II},\ \mathrm{III},\ \mathrm{IV},\ \mathrm{II}^*,\ \mathrm{III}^*,\ \mathrm{IV}^*,\ \mathrm{I}_0^*(j \neq 0, 1728),\ \mathrm{I}_0^*(j=0),\ \mathrm{I}_0^*(j=1728)\Big\}.$$

At the cusp $j = \infty$, the inertia label records only the cusp shape ($\mathrm{I}_\bullet$ or $\mathrm{I}_\bullet^*$); the additional integer $k \geq 1$ (contact order with the boundary, equivalently the pole order of j) is *not* part of the local twisting conditions Γ of [BPS22, Def. 3.1]: for I_k one has $(r, a) = (0, 0)$ so no stacky marking appears, while for I_k^* one has $(r, a) = (2, 1)$ independently of k. The contact order is instead determined by the discriminant valuation of the Weierstrass model, equivalently the pole order of the j-map at $j = \infty$ ([BPS22, Thm. 7.12]). In the generating function $\mathcal{H}(s; \mathbf{x})$, the contact order becomes a free summation variable, collapsed by geometric resummation (Lemma 2.4). Accordingly we set

$$\mathcal{A} \;\coloneqq\; \mathcal{A}_{\mathrm{nc}} \;\sqcup\; \{\mathrm{I}_\bullet,\ \mathrm{I}_\bullet^*\}.$$

For $\alpha \in \mathcal{A}_{\mathrm{nc}}$, let $m(\alpha) \in \mathbb{Z}_{\geq 1}$ be the number of irreducible components of the corresponding Kodaira fiber, so that $m(\alpha) - 1$ is its contribution to the trivial lattice. For the cusp shapes $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$, the component number depends on the contact order $k \geq 1$; this k-dependence is again incorporated by geometric resummation. Note that the same Kodaira symbol may correspond to distinct inertia components; for example, I_0^* splits according to whether $j \in \{0\}$, $j \in \{1728\}$ or $j \notin \{0, 1728\}$.

Definition 2.3. Fix an auxiliary variable s with $s^{12} = t$. Introduce variables $\{x_\alpha\}_{\alpha\in\mathcal{A}}$ and define

$$\mathcal{H}(s;\mathbf{x}) \coloneqq \sum_{n\geq 0}\sum_{\mathfrak{f}}\Big(\prod_{\nu\in\mathfrak{f}} x_{\alpha_\nu}\Big)\big\{\mathcal{W}_n^{\min,(\mathfrak{f})}\big\}s^{12n} \in K_0(\mathrm{Stck}_k)[\mathbf{x}][\![s]\!], \tag{6}$$

where for fixed n the inner sum ranges over the finitely many geometric fiber configurations $\mathfrak{f}$ occurring in height n.

For each singular fiber F_ν in $\mathfrak{f}$, let $\alpha_\nu \in \mathcal{A}$ denote the corresponding inertia/evaluation label. Away from the cusp $j = \infty$ this label is the Kodaira symbol, while over $j = \infty$ it records only the cusp shape $\mathrm{I}_\bullet$ or $\mathrm{I}^*_\bullet$. The additional contact order $k \geq 1$ at the cusp is *not* recorded by the variables x_α.

Kapranov zeta function. For a k–variety (or Deligne–Mumford stack) X, we write

$$\zeta_X(y) \coloneqq \sum_{N\geq 0}\big\{\mathrm{Sym}^N(X)\big\}\,y^N \in K_0(\mathrm{Stck}_k)[\![y]\!]$$

for the Kapranov motivic zeta function [Kap00]. For $X = \mathbb{P}^1$ one has $\zeta_{\mathbb{P}^1}(y) = 1/\big((1-y)(1-\mathbb{L}\,y)\big)$.

2.2. **Cusp resummation and motivic transversality.**

Lemma 2.4. *Let R be a commutative ring.*

(1) Geometric resummation *Fix $A \in R$. For integers $a, c \geq 1$ and $b, d \in \mathbb{Z}$, one has in $R[\![u,t]\!]$*

$$\sum_{k\geq 1} A u^{ak+b}t^{ck+d} = A u^{a+b}t^{c+d}\cdot\frac{1}{1-u^a t^c}. \tag{7}$$

Moreover, if $k_1,\dots,k_M \geq 1$ are independent and contribute multiplicatively with the same step (a,c), then

$$\sum_{k_1,\dots,k_M\geq 1} A\prod_{i=1}^{M} u^{ak_i+b}t^{ck_i+d} = A\Big(u^{a+b}t^{c+d}\Big)^M\cdot\frac{1}{(1-u^a t^c)^M}. \tag{8}$$

Equivalently, each marking contributes one factor $(1-u^a t^c)^{-1}$, so M such markings contribute the power $(1-u^a t^c)^{-M}$, up to the monomial shift $\big(u^{a+b}t^{c+d}\big)^M$.

(2) Cusp shapes for Z_{Triv} *Assume* $\mathrm{char}(k) \neq 2,3$ *and work in* $R = K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$. *Introduce an auxiliary variable s with $t = s^{12}$, so that t^n corresponds to* $\deg(\Delta) = 12n$, *while s records the integral discriminant degree* $\deg(\Delta)$.

After specializing $x_\beta = u^{m(\beta)-1}$ for $\beta \in \mathcal{A}_{\mathrm{nc}}$, a cusp marking of shape $\mathrm{I}_\bullet$ (resp. $\mathrm{I}^*_\bullet$) with contact order $k \geq 1$ contributes weight $u^{k-1}s^k$ (resp. $u^{k+4}s^{k+6}$), since

$$m(\mathrm{I}_k)-1 = k-1, \qquad \nu(\Delta) = k, \qquad m(\mathrm{I}^*_k)-1 = k+4, \qquad \nu(\Delta) = k+6.$$

Hence summing over $k \geq 1$ at a single cusp marking gives, in $R[\![u,s]\!]$,

$$x_{\mathrm{I}_\bullet} = \sum_{k\geq 1} u^{k-1}s^k = \frac{s}{1-us}, \qquad x_{\mathrm{I}^*_\bullet} = \sum_{k\geq 1} u^{k+4}s^{k+6} = \frac{u^5 s^7}{1-us}. \tag{9}$$

In particular, each cusp marking of either shape contributes one factor $(1-us)^{-1}$ *after resummation. Thus a factor type* j *with* $\beta_{j,\mathrm{I}_\bullet}$ *markings of shape* $\mathrm{I}_\bullet$ *and* $\beta_{j,\mathrm{I}_\bullet^*}$ *markings of shape* $\mathrm{I}_\bullet^*$ *contributes the cusp factor*

$$(1-us)^{-(\beta_{j,\mathrm{I}_\bullet}+\beta_{j,\mathrm{I}_\bullet^*})},$$

together with the monomial shift

$$u^{5\beta_{j,\mathrm{I}_\bullet^*}}\, s^{\beta_{j,\mathrm{I}_\bullet}+7\beta_{j,\mathrm{I}_\bullet^*}}$$

coming from (9).

Proof. For (7), factor out the $k=1$ term and sum the geometric series. Equation (8) follows by independence of the k_i. Part (2) follows by summing the geometric series $\sum_{k\geq 1} u^{k-1}s^k = s/(1-us)$ and $\sum_{k\geq 1} u^{k+4}s^{k+6} = u^5s^7/(1-us)$. ∎

The multiplicative singular fibers I_k for $k\geq 1$ impose no vanishing conditions on (a_4,a_6); their positions and contact orders are determined entirely by the discriminant $\Delta = 4a_4^3+27a_6^2$. We isolate the local motivic class of the discriminant condition at a single point.

Definition 2.5. For each $\ell\geq 1$, the *local multiplicative locus* is

$$\Lambda(\ell) \coloneqq \big\{(\alpha_0,\dots,\alpha_\ell,\ \beta_0,\dots,\beta_\ell)\in \mathbb{A}^{\ell+1}\times\mathbb{A}^{\ell+1} \;\big|\; \mathrm{ord}_0(4A^3+27B^2)=\ell,\ (\alpha_0,\beta_0)\neq(0,0)\big\},$$

where $A=\sum_{j=0}^{\ell}\alpha_j z^j$ and $B=\sum_{j=0}^{\ell}\beta_j z^j$. The class $\{\Lambda(\ell)\}\in K_0(\mathrm{Var}_k)$ depends only on ℓ, not on n or the choice of base point in $\mathbb{P}^1$.

The nonlinearity of $\Delta = 4a_4^3+27a_6^2$ means that prescribing $\mathrm{ord}_{\nu_i}(\Delta)=k_i$ at r distinct points requires controlling the k_i-jets of (a_4,a_6) at each ν_i. The resulting conditions are independent whenever $\sum_i(k_i+1)\leq 4n+1=\dim H^0(\mathcal{O}(4n))$.

Lemma 2.6. *Let* k *be a perfect field with* $\mathrm{char}(k)\neq 2,3$. *Fix* r *distinct closed points* $\nu_1,\dots,\nu_r\in\mathbb{P}^1$ *and positive integers* $k_1,\dots,k_r\geq 1$. *For* $n\geq 1$, *define the* multiplicative multi-point locus $\mathcal{M}_n(k_\bullet,\nu_\bullet)\coloneqq$

$$\big\{(a_4,a_6)\in H^0\big(\mathcal{O}(4n)\big)\oplus H^0\big(\mathcal{O}(6n)\big) \;\big|\; \mathrm{ord}_{\nu_i}(4a_4^3+27a_6^2)=k_i,\ (a_4(\nu_i),a_6(\nu_i))\neq(0,0)\ \ \forall\, i\big\}$$

and the jet evaluation map

$$\mathrm{ev}\colon H^0\big(\mathcal{O}(4n)\big)\oplus H^0\big(\mathcal{O}(6n)\big)\longrightarrow \bigoplus_{i=1}^{r}\Big(J^{k_i}_{\nu_i}\big(\mathcal{O}(4n)\big)\ \oplus\ J^{k_i}_{\nu_i}\big(\mathcal{O}(6n)\big)\Big).$$

If $\sum_{i=1}^{r}(k_i+1)\leq 4n+1$, *then* ev *is surjective and*

$$\big\{\mathcal{M}_n(k_\bullet,\nu_\bullet)\big\} \;=\; \mathbb{L}^{\,10n+2-\sum_i 2(k_i+1)}\cdot\prod_{i=1}^{r}\big\{\Lambda(k_i)\big\}. \tag{10}$$

Proof. The condition $\mathrm{ord}_{\nu_i}(\Delta)=k_i$ depends only on the k_i-jet of (a_4,a_6) at ν_i, since $\Delta^{(j)}(\nu_i)$ for $j\leq k_i$ is determined by $\{a_4^{(\ell)}(\nu_i),a_6^{(\ell)}(\nu_i)\}_{\ell\leq j}$ by the Leibniz rule. The jet evaluation map ev decomposes as the direct sum of Hermite interpolation maps for a_4 and a_6 separately. The former is surjective when $\sum_i(k_i+1)\leq 4n+1=\dim H^0(\mathcal{O}(4n))$, the latter when $\sum_i(k_i+1)\leq 6n+1=\dim H^0(\mathcal{O}(6n))$, so ev is surjective under the stated bound. Since ev is a surjective linear map with kernel $\mathbb{A}^{10n+2-\sum_i 2(k_i+1)}$, the vector bundle relation gives $\{\mathrm{ev}^{-1}(Z)\}=\mathbb{L}^{\dim\ker}\cdot\{Z\}$ for any

constructible Z in the target. The conditions at distinct points v_i act on disjoint jet factors, so $Z = \prod_i \Lambda(k_i)$ and (10) follows. ∎

In the multiplicative sector, $\sum_i k_i = 12n - D_{\mathrm{add}}$ where D_{add} is the additive discriminant degree, so the surjectivity bound fails whenever the multiplicative fibers carry more than roughly $\frac{1}{3}$ of $12n$. For the generic semistable elliptic surface all singular fibers are multiplicative and the bound fails at every height.

For a fixed trivial lattice rank m, the *visible* fibers (I_k with $k \geq 2$, I_k^* with $k \geq 1$, and the additive types with $m_v - 1 \geq 1$) impose finitely many jet conditions whose total number is bounded independently of n. The jet evaluation map at the visible support points is therefore surjective for n sufficiently large (depending on m). The residual I_1 fibers (invisible, with $m_v - 1 = 0$) do not enter the jet evaluation map; their contribution is handled by the motivic discriminant stabilization.

2.3. **Euler product decomposition.** We now decompose the multivariate height series into an Euler product over local factor types. Let J denote the finite set of local factor types from Definition 2.2. For each $j \in J$, write $A_j \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$ for the normalized one-fiber motivic class, $c_j \geq 0$ for the discriminant degree increment, and $\beta_{j,\alpha} \in \mathbb{Z}_{\geq 0}$ for the number of markings of inertia type $\alpha \in \mathcal{A}$ in a factor of type j. For non-cusp types, $c_j \geq 1$ is the fixed discriminant valuation of the fiber; for the cusp types $\mathrm{I}_\bullet$ and $\mathrm{I}_\bullet^*$, the discriminant degree depends on the contact order k and is encoded in the cusp variables $x_{\mathrm{I}_\bullet}$, $x_{\mathrm{I}_\bullet^*}$ via Lemma 2.4 rather than in a fixed exponent, so we set $c_j = 0$. Define

$$Y_j(s;\mathbf{x}) := A_j\Big(\prod_{\alpha\in\mathcal{A}} x_\alpha^{\beta_{j,\alpha}}\Big)s^{c_j}.$$

Lemma 2.7. *After inverting $\mathbb{L}$, the multivariate height series $\mathcal{H}(s;\mathbf{x})$ of (6) satisfies*

$$\mathcal{H}(s;\mathbf{x}) = \prod_{j\in J}\big(1 - Y_j(s;\mathbf{x})\big)^{-\{\mathbb{P}^1\}} + P(s;\mathbf{x}), \tag{11}$$

where

$$(1-Y_j)^{-\{\mathbb{P}^1\}} = \sum_{N\geq 0}\{\mathrm{Sym}^N(\mathbb{P}^1)\}\,Y_j^N = \frac{1}{(1-Y_j)(1-\mathbb{L}\,Y_j)}$$

is the Kapranov zeta function evaluated at Y_j, and $P(s;\mathbf{x}) := \mathcal{H}(s;\mathbf{x}) - \prod_{j\in J}(1 - Y_j)^{-\{\mathbb{P}^1\}}$ is the remainder. For each monomial in the additive variables $\{x_\alpha\}_{\alpha\in\mathcal{A}_{\mathrm{nc}}}$, the coefficient of the Euler product is rational in s. The remainder P absorbs the small-height corrections, the unresolved multiplicative sector, and the cusp contact orders; its s-structure is resolved by the motivic discriminant stabilization in Theorem 2.12.

Proof. The weighted-linear-series / twisted-maps correspondence of [BPS22, Thm. 3.3 & Prop. 5.8] gives a finite locally closed stratification of the height moduli stack $\mathcal{W}_n^{\min}$ into charts $\mathcal{H}^{\Gamma}_{d,\mathbb{P}^1}(\mathcal{P}(4,6),\mathcal{O}(1))/S_\Gamma$ indexed by admissible local conditions Γ and stable height d with $n = d + \sum a_i/r_i$.

For each additive type $j \in J_{\mathrm{add}}$, the one-fiber stratum $\mathcal{W}^{\gamma_j}_{n,\mathbb{P}^1}$ is a Zariski-locally trivial fibration over $\mathbb{P}^1$ by [BPS22, Prop. 6.7], with motivic class $(\mathbb{L}^2-1)\mathbb{L}^{10n-p_j-q_j}$ for $n \gg 0$. Normalizing by $\{\mathbb{P}^1\}\cdot\mathbb{L}^{10n}$ gives the n-independent coefficient $A_j = (\mathbb{L}-1)\mathbb{L}^{-p_j-q_j}$. For the multiplicative type $\mathrm{I}_\bullet$, the coefficient $A_{\mathrm{I}_k} = \mathbb{L}^{18}$ is computed

in [HP19, Cor. 2]; the formula $A_j = (\mathbb{L}-1)\mathbb{L}^{-p_j-q_j}$ does not apply since $(p_j, q_j) = (0,0)$ and the discriminant condition is nonlinear. For I_k^*, the additive vanishing $(p,q) = (2,3)$ enters through J_{add} while the cusp contact order k is deferred to the cusp variable $x_{\mathrm{I}_\bullet^*}$.

Since I_k fibers have $(p_j, q_j) = (0,0)$, they impose no vanishing conditions on (a_4, a_6); their positions and contact orders are determined entirely by $\Delta = 4a_4^3 + 27a_6^2$. Multi-point independence of discriminant root multiplicities requires $\sum_i (k_i + 1) \leq 4n+1$ by Lemma 2.6, which fails in the multiplicative-heavy regime. At the level of this lemma, the multiplicative contribution is encoded formally by the cusp variables and the remainder P absorbs the unresolved s-structure.

We decompose $\mathcal{H}(s;\mathbf{x}) = \sum_T \mathcal{H}_T(s;\mathbf{x}_{\mathrm{mult}}) \cdot \prod_{j\in J_{\mathrm{add}}} x_j^{N_j}$ by additive configuration $T = (N_j)_{j\in J_{\mathrm{add}}}$. Fix T. The additive vanishing conditions at $|T| = \sum_j N_j$ distinct points of $\mathbb{P}^1$ impose $P(T) = \sum_j N_j p_j$ conditions on $H^0(\mathcal{O}(4n))$ and $Q(T) = \sum_j N_j q_j$ on $H^0(\mathcal{O}(6n))$. These are independent whenever $P(T) \leq 4n+1$ and $Q(T) \leq 6n+1$. Since $P(T)$ and $Q(T)$ depend only on T while $\dim H^0(\mathcal{O}(4n))$ and $\dim H^0(\mathcal{O}(6n))$ grow linearly in n, there exists $n_1(T)$ such that independence holds for all $n \geq n_1(T)$.

For $n \geq n_1(T)$, the motivic class of the locus in U_n° with additive configuration T at $|T|$ unordered distinct points of $\mathbb{P}^1$ factors as a product of independent local contributions A_j, yielding the Euler product for the additive component of $\mathcal{H}_T$. The Kapranov identity $(1-Y_j)^{-\{\mathbb{P}^1\}} = 1/((1-Y_j)(1-\mathbb{L}Y_j))$ converts each factor into a rational function of s. For each fixed monomial $\prod_{\alpha\in\mathcal{A}_{\mathrm{nc}}} x_\alpha^{N_\alpha}$, there is a unique additive configuration $T = (N_\alpha)_{\alpha\in\mathcal{A}_{\mathrm{nc}}}$, so only finitely many T contribute to any given monomial and the coefficient is rational in s.

Summing over T and absorbing the small-height corrections $n < n_1(T)$ (finitely many, hence polynomial in s), the unresolved multiplicative sector (simple roots of $\Delta = 4a_4^3 + 27a_6^2$ at coprime points, handled by the motivic discriminant stabilization of Proposition 2.11), and the cusp contact orders (handled by the geometric resummation of Lemma 2.4 after specialization) into $P(s;\mathbf{x})$ gives (11). ∎

Under the specialization $x_\alpha = u^{m(\alpha)-1}$, two types of invisible fibers acquire unbounded multiplicities at fixed u^m: type II (with $m(\mathrm{II}) - 1 = 0$, so $x_{\mathrm{II}} \mapsto 1$) and type I_1 (with $m(\mathrm{I}_1) - 1 = 0$). For type II, the multiplicity is controlled on $\mathcal{W}_n^\circ$ by $N_{\mathrm{II}} \leq \deg(a_4) = 4n$ (Proposition 2.9(2)), while on the isotrivial loci separate exact Euler products apply (Proposition 2.8). For type I_1, the residual simple roots of $\Delta = 4a_4^3 + 27a_6^2$ are handled by the motivic discriminant stabilization (Proposition 2.11).

Proposition 2.8. *Let k be a perfect field with* $\mathrm{char}(k) \neq 2,3$. *Set $s = t^{1/12}$.*

(1) The $j \equiv 0$ locus. *On $\{a_4 = 0\}$, the Weierstrass model reduces to $y^2 = x^3 + a_6(t)$ with $a_6 \in H^0(\mathbb{P}^1, \mathcal{O}(6n)) \smallsetminus \{0\}$ subject to the minimality constraint $v_v(a_6) < 6$ for all v. The discriminant is $\Delta = 27\,a_6^2$, so each singular fiber at v is determined by*

$v_v(a_6) \in \{1,2,3,4,5\}$:

$v_v(a_6)$	1	2	3	4	5
Fiber type	II	IV	I_0^* (j=0)	IV*	II*
$m_v - 1$	0	2	4	6	8

Since all local conditions are linear *on the single coefficient space $H^0(\mathcal{O}(6n))$, the local-to-global factorization is exact at all heights: no sparsity threshold or surjectivity condition is needed. Define $A_v^{(0)} := (\mathbb{L}-1)\mathbb{L}^{-v}$ for $v \in \{1,\dots,5\}$, and let $w(v) := m_v - 1$ be the trivial lattice contribution from the table above (explicitly $w(1,\dots,5) = 0,2,4,6,8$). Then*

$$Z_{\mathrm{Triv}}^{j=0}(u;t) \;=\; u^2 \cdot \prod_{v=1}^{5} \zeta_{\mathbb{P}^1}\big(A_v^{(0)}\, u^{w(v)} s^{2v}\big). \tag{12}$$

(2) The $j \equiv 1728$ locus. *On $\{a_6 = 0\}$, the model reduces to $y^2 = x^3 + a_4(t)x$ with $a_4 \in H^0(\mathbb{P}^1, \mathcal{O}(4n)) \smallsetminus \{0\}$ and $v_v(a_4) < 4$ for all v. The discriminant is $\Delta = 4\,a_4^3$, and*

$v_v(a_4)$	1	2	3
Fiber type	III	I_0^* (j=1728)	III*
$m_v - 1$	1	4	7

All conditions are linear on $H^0(\mathcal{O}(4n))$, and the factorization is again exact:

$$Z_{\mathrm{Triv}}^{j=1728}(u;t) \;=\; u^2 \cdot \prod_{\mu=1}^{3} \zeta_{\mathbb{P}^1}\big(A_\mu^{(1728)}\, u^{w'(\mu)} s^{3\mu}\big), \tag{13}$$

with $A_\mu^{(1728)} = (\mathbb{L}-1)\mathbb{L}^{-\mu}$ and $w'(1) = 1$, $w'(2) = 4$, $w'(3) = 7$.

Both (12) *and* (13) *are finite products of evaluations of $\zeta_{\mathbb{P}^1}(y) = 1/\big((1-y)(1-\mathbb{L}\,y)\big)$ at monomials, hence strictly rational in s. Since the discriminant is a pure power of a single Weierstrass coefficient on both loci, every singular fiber is additive and neither the cusp resummation (Lemma 2.4) nor the motivic transversality (Lemma 2.6) is needed.*

Proof. We prove (1); part (2) is identical with $H^0(\mathcal{O}(6n))$ replaced by $H^0(\mathcal{O}(4n))$.

With $a_4 = 0$, the moduli at height n is $H^0(\mathbb{P}^1, \mathcal{O}(6n)) \smallsetminus \{0\}$ modulo $\mathbb{G}_m$, and each local condition $v_v(a_6) \geq v$ is codimension-v and linear on $H^0(\mathcal{O}(6n))$. For distinct points $v_1, \dots, v_r$, these conditions are independent whenever the Hermite interpolation map $H^0(\mathcal{O}(6n)) \to \bigoplus_i J_{v_i}^{v_i-1}(\mathcal{O}(6n))$ is surjective, which requires $\sum_i v_i \leq 6n+1 = \dim H^0(\mathcal{O}(6n))$. Since minimality forces $v_i \leq 5$ and $\sum_i 2v_i = 12n$ (discriminant degree), one has $\sum_i v_i = 6n < 6n+1$, so surjectivity holds at all heights and the Euler product is exact with no correction. The local factor coefficient $A_v^{(0)} = (\mathbb{L}-1)\mathbb{L}^{-v}$ records the codimension-v vanishing with nonzero leading term. ∎

Proposition 2.9. *Decompose the moduli as a disjoint union*

$$\mathcal{W}_n^{\min} \;=\; \mathcal{W}_n^{\circ} \;\sqcup\; \mathcal{W}_n^{j=0} \;\sqcup\; \mathcal{W}_n^{j=1728},$$

where $\mathcal{W}_n^{j=0}$ parametrizes $j \equiv 0$ (equivalently $a_4 \equiv 0$), $\mathcal{W}_n^{j=1728}$ parametrizes $j \equiv 1728$ (equivalently $a_6 \equiv 0$), and $\mathcal{W}_n^\circ$ is the complement (equivalently $a_4 \not\equiv 0$ and $a_6 \not\equiv 0$; this includes all non-constant j-invariant fibrations as well as constant $j \notin \{0, 1728\}$).

(1) *On the isotrivial loci, the Euler products of Proposition 2.8 are exact at all heights, since all local conditions are linear on a single coefficient space.*
(2) *On $\mathcal{W}_n^\circ$, every additive fiber at v requires $\nu_v(a_4) \geq 1$. Since $a_4 \not\equiv 0$ has degree $\leq 4n$, the total additive jet codimension on $H^0(\mathcal{O}(4n))$ is at most $4n$, which is strictly less than $\dim H^0(\mathcal{O}(4n)) = 4n+1$. The analogous bound holds for a_6. Hence the Hermite interpolation maps for the additive local conditions are surjective at all heights on $\mathcal{W}_n^\circ$, and the sparsity argument of Lemma 2.7 applies to the additive sector. The multiplicative sector (root multiplicities of $\Delta = 4a_4^3 + 27a_6^2$) is not covered by this surjectivity and is handled separately by the motivic discriminant stabilization of Proposition 2.11.*
(3) *At $u = 1$ the three loci assemble to the unweighted height zeta function $Z_{\vec{\lambda}}(t)$, which equals the exact Euler product of* [BPS22, Thm. 8.9].

Proof. Part (1) is Proposition 2.8. For (2), $\sum_v \nu_v(a_4) \leq \deg(a_4) = 4n < 4n+1 = \dim H^0(\mathcal{O}(4n))$ and $\sum_v \nu_v(a_6) \leq 6n < 6n+1 = \dim H^0(\mathcal{O}(6n))$, so the Hermite interpolation maps for the additive conditions are surjective at all heights. Part (3) follows from [BPS22, Thm. 8.9]. ∎

2.4. **Motivic discriminant stabilization.** The motivic transversality (Lemma 2.6) establishes multi-point independence of discriminant conditions when the jet evaluation map is surjective: $\sum_i (k_i + 1) \leq 4n + 1$. For a purely multiplicative fibration at height n, the total number of evaluation conditions is $\sum_i (k_i + 1) = 12n + r \geq 12n + 1$, which exceeds the surjectivity bound $4n + 1$ for every $n \geq 1$. More generally, whenever the multiplicative fibers account for more than a third of the discriminant degree, the surjectivity condition fails. Since the generic elliptic surface over $\mathbb{P}^1$ is semistable, having all singular fibers multiplicative, the motivic transversality alone does not cover the dominant part of moduli. To handle this regime, we adapt the motivic stabilization method of Vakil–Wood [VW15] to the discriminant $\Delta = 4a_4^3 + 27a_6^2$ of the Weierstrass model. The key idea is a truncated inclusion–exclusion in the dimensional completion of the Grothendieck ring, where the error from discriminant root multiplicities becomes dimensionally negligible.

Recall [VW15, Section 1.4] that the *dimensional filtration* on $\mathcal{M}_{\mathbb{L}} := K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$ is defined by

$$F^d \mathcal{M}_{\mathbb{L}} := \left\langle [X]\mathbb{L}^{-j} : \dim X - j \leq -d \right\rangle.$$

Let $\widehat{\mathcal{M}_{\mathbb{L}}}$ denote the completion of $\mathcal{M}_{\mathbb{L}}$ with respect to this filtration. This is Kontsevich's ring of motivic integrals [Kon95]; it inherits a ring structure [VW15, §1.4]. A sequence $a_N \to 0$ in $\widehat{\mathcal{M}_{\mathbb{L}}}$ means $\dim(a_N) \to -\infty$.

For the u-extraction, we call a singular fiber *visible* if $m(F_v) - 1 \geq 1$, i.e. if it contributes positively to the trivial lattice beyond the minimal value 2. On $\mathcal{W}_n^\circ$, the visible fiber types are: the additive types III, IV, I_0^*, IV*, III*, II* (each with $m_v - 1 \geq 1$); the multiplicative types I_k with $k \geq 2$ (with $m_v - 1 = k - 1 \geq 1$); and

the starred cusp types I_k^* with $k \geq 1$ (with $m_v - 1 = k + 4 \geq 5$). The *invisible* types are II (with $m_v - 1 = 0$) and I_1 (with $m_v - 1 = 0$).

For a fixed power u^m in Z_{Triv}, the visible fibers have total weight $\sum(m_v - 1) = m-2$. Since each visible fiber contributes weight ≥ 1, the number of visible fibers is at most $m-2$. Their types and (unordered) positions on $\mathbb{P}^1$ form a *visible configuration* σ, with weight $|\sigma|_{\mathrm{wt}} := \sum_{v\in\sigma}(m_v - 1) = m-2$; there are finitely many such σ for each m.

Fix a visible configuration σ with $|\sigma|_{\mathrm{wt}} = m-2$. Each visible fiber imposes jet conditions at its support point: an additive visible fiber of type Θ requires $\nu_v(a_4) \geq p_\Theta$ and $\nu_v(a_6) \geq q_\Theta$, while a multiplicative visible fiber I_{k_i} (with $k_i \geq 2$) or the cusp component of $\mathrm{I}_{k_j}^*$ requires prescribing (k_i+1) or (k_j+1) jets of (a_4, a_6) respectively. Write

$$D_{a_4}(\sigma) := \sum_{v\in\sigma_{\mathrm{add}}} p_{\Theta_v} + \sum_{v\in\sigma_{\mathrm{mult}}} (k_v+1), \qquad D_{a_6}(\sigma) := \sum_{v\in\sigma_{\mathrm{add}}} q_{\Theta_v} + \sum_{v\in\sigma_{\mathrm{mult}}} (k_v+1)$$

for the total jet conditions imposed on $H^0(\mathcal{O}(4n))$ and $H^0(\mathcal{O}(6n))$ respectively, and $D(\sigma) := D_{a_4}(\sigma) + D_{a_6}(\sigma)$. Then

$$D(\sigma) \leq D(m) := \max_{\sigma'\in\Sigma(m)} D(\sigma') < \infty, \tag{14}$$

since $\Sigma(m)$ is finite: each visible fiber contributes weight ≥ 1 to the total $m-2$, bounding the number of visible fibers, while the weight constraint $\sum(k_i - 1) \leq m-2$ bounds the total multiplicative jet demand, and the additive jet demands are bounded by the fiber types in σ.

By Lemma 2.6 (for the multiplicative visible fibers) and the sparsity argument (Lemma 2.7, for the additive visible fibers), the jet evaluation map at the σ-support points is surjective for $n \geq n_0(m) := \max_{\sigma\in\Sigma(m)}\big(\lceil (D_{a_4}(\sigma)-1)/4\rceil, \lceil (D_{a_6}(\sigma)-1)/6\rceil\big)$. Let

$$U_n^\circ := \big\{(a_4,a_6) \in H^0\big(\mathcal{O}(4n)\big) \oplus H^0\big(\mathcal{O}(6n)\big) : a_4 \not\equiv 0,\ a_6 \not\equiv 0\big\}$$

denote the locus where $j \not\equiv 0$ and $j \not\equiv 1728$, and define

$$G_\sigma(n) := \big\{(a_4,a_6) \in U_n^\circ : \text{visible conditions of } \sigma \text{ hold at unordered support points}\big\}.$$

For $n \geq n_0(m)$, the motivic class of $G_\sigma(n)$ satisfies

$$\big\{G_\sigma(n)\big\} = C(\sigma)\cdot \mathbb{L}^{10n+2-D(\sigma)} + E_\sigma(n), \tag{15}$$

where $C(\sigma) \in \mathcal{M}_{\mathbb{L}}$ depends on σ alone and $\dim(E_\sigma(n)) \leq 6n+1-D_{a_6}(\sigma)$ accounts for the removal of the isotrivial loci from the kernel.

The locus $G_\sigma(n)$ includes Weierstrass data whose exact visible configuration is a strict enlargement $\sigma' \supsetneq \sigma$: the data may have extra visible fibers at points outside the σ-support. Such data have $T(S) > m$ and contribute to $u^{m'}$ with $m' > m$, not to $[u^m]Z_{\mathrm{Triv}}$. To extract the exact visible locus

$$F_\sigma(n) := \big\{(a_4,a_6) \in G_\sigma(n) : \text{no visible fibers outside the } \sigma\text{-support}\big\},$$

we must remove the extra-visible-fiber contributions. On the coprime locus outside the σ-support, the condition "no visible multiplicative fiber at v" is $\mathrm{ord}_v(\Delta) \leq 1$ (i.e., Δ has at most a simple root at v). Equivalently, the residual discriminant

must have no multiple roots on $\mathbb{P}^1 \smallsetminus V_\sigma$, where V_σ is the σ-support. This is a motivic discriminant-squarefreeness condition on the image of the nonlinear map $\Delta = 4a_4^3 + 27a_6^2$, which we handle by the Vakil–Wood method. (Extra visible additive fibers cannot occur on the coprime locus outside V_σ: every additive type has $v_v(a_4) \geq 1$ and $v_v(a_6) \geq 1$, forcing $(a_4(v), a_6(v)) = (0,0)$.)

We first compute the codimension of the extra multiple root condition at a single coprime point.

Lemma 2.10. *Let $v \in \mathbb{P}^1 \smallsetminus V_\sigma$. On the coprime locus $\{(a_4(v), a_6(v)) \neq (0,0)\}$, the condition $\mathrm{ord}_v(\Delta) \geq 2$ defines a constructible subset of the 1-jet space $J_v^1(\mathcal{O}(4n)) \oplus J_v^1(\mathcal{O}(6n)) \cong \mathbb{A}^4$ of class $\mathbb{L}^2 - \mathbb{L}$ and codimension 2.*

Proof. Write $\alpha = a_4(v)$, $\alpha' = a_4'(v)$, $\beta = a_6(v)$, $\beta' = a_6'(v)$ for the 1-jet coordinates. The condition $\mathrm{ord}_v(\Delta) \geq 2$ on the coprime locus $(\alpha, \beta) \neq (0,0)$ consists of two parts. First, $\Delta(v) = 4\alpha^3 + 27\beta^2 = 0$ restricts (α, β) to the cuspidal cubic $\{4x^3 + 27y^2 = 0\} \smallsetminus \{(0,0)\}$, which is parametrized by $(\alpha, \beta) = (-3\lambda^2, 2\lambda^3)$ for $\lambda \in \mathbb{G}_m$; its class is $\mathbb{L} - 1$. Second, $\Delta'(v) = 12\alpha^2\alpha' + 54\beta\beta' = 0$: since $4\alpha^3 + 27\beta^2 = 0$ with $(\alpha, \beta) \neq (0,0)$ forces $\beta \neq 0$, this is a nontrivial linear condition on (α', β'), cutting codimension 1 in $\mathbb{A}^2$; its solution space has class $\mathbb{L}$. The total class in $\mathbb{A}^4$ is $(\mathbb{L}-1)\cdot\mathbb{L} = \mathbb{L}^2 - \mathbb{L}$, which has dimension 2 inside the ambient $\mathbb{A}^4$. The codimension is 2. ∎

Following [VW15, Section 3], we perform a truncated inclusion–exclusion to pass from $G_\sigma(n)$ to $F_\sigma(n)$. For an unordered set $\tau = \{w_1, \ldots, w_\ell\}$ of ℓ distinct points in $\mathbb{P}^1 \smallsetminus V_\sigma$, let $G_{\sigma,\tau}(n) \subset G_\sigma(n)$ denote the sublocus where $\mathrm{ord}_{w_i}(\Delta) \geq 2$ for each $w_i \in \tau$. For $k \geq 0$, let $G_{\sigma,\geq k}(n)$ denote the sublocus of $G_\sigma(n)$ where the residual discriminant has at least k additional multiple roots (each of multiplicity ≥ 2) on $\mathbb{P}^1 \smallsetminus V_\sigma$. The standard inclusion–exclusion on the labeled bad-point conditions gives, at truncation depth N,

$$F_\sigma(n) \;=\; \sum_{\ell=0}^{N} (-1)^\ell \sum_{|\tau|=\ell} \big\{G_{\sigma,\tau}(n)\big\} \;+\; (-1)^{N+1} \big\{G_{\sigma,\geq N+1}(n)\big\}.$$

For each $\ell \leq N$, the locus $G_{\sigma,\tau}(n)$ parametrizes coprime data satisfying the σ-conditions and having $\mathrm{ord}_{w_i}(\Delta) \geq 2$ at the ℓ points $w_1, \ldots, w_\ell$ of τ. The 1-jet evaluation map at the σ-support and the τ-points,

$$\mathrm{ev}\colon H^0(\mathcal{O}(4n)) \oplus H^0(\mathcal{O}(6n)) \longrightarrow \bigoplus_{v \in V_\sigma} \big(J_v^{k_v} \oplus J_v^{k_v}\big) \;\oplus\; \bigoplus_{i=1}^{\ell} \big(J_{w_i}^1 \oplus J_{w_i}^1\big),$$

has source dimension $10n + 2$ and target dimension $D(\sigma) + 4\ell$. This is surjective whenever

$$D_{a_4}(\sigma) + 2\ell \;\leq\; 4n + 1 \qquad \text{and} \qquad D_{a_6}(\sigma) + 2\ell \;\leq\; 6n + 1. \tag{16}$$

When surjective, Lemma 2.10 gives the factorization

$$\sum_{|\tau|=\ell} \big\{G_{\sigma,\tau}(n)\big\} \;=\; C(\sigma) \cdot \big\{\mathrm{Sym}^\ell(\mathbb{P}^1 \smallsetminus V_\sigma)\big\} \cdot (\mathbb{L}^2 - \mathbb{L})^\ell \cdot \mathbb{L}^{10n+2-D(\sigma)-4\ell}, \tag{17}$$

where the first factor is the local class at the σ-points (same as in (15)), the second chooses the ℓ extra positions, the third is the local multiple-root class from

Lemma 2.10 at each extra point, and the fourth is the kernel dimension. Here and below, all factorizations on U_n° hold up to the isotrivial correction of (15), which is absorbed by the dimensional completion.

Proposition 2.11. *For each visible configuration σ with $|\sigma|_{\mathrm{wt}} = m-2$, and each truncation depth $N \geq 0$, the truncated inclusion–exclusion*

$$F_\sigma^{(N)}(n) \coloneqq \sum_{\ell=0}^{N} (-1)^\ell \sum_{|\tau|=\ell} \{G_{\sigma,\tau}(n)\} \tag{18}$$

satisfies the following for all $n \geq n_0(\sigma,N) \coloneqq \max\big(\lceil (D_{a_4}(\sigma)+2N+1)/4 \rceil, \lceil (D_{a_6}(\sigma)+2N+1)/6 \rceil\big)$, where $D_{a_4}(\sigma)+D_{a_6}(\sigma) = D(\sigma) \leq D(m)$:

(1) *Each summand in* (18) *is given by* (17).

(2) *The truncation error satisfies the dimension bound*

$$\dim\big(F_\sigma(n) - F_\sigma^{(N)}(n)\big) \;\leq\; 10n+2-D(\sigma)-(N+1). \tag{19}$$

(3) *After normalizing by $\mathbb{L}^{10n+2-D(\sigma)}$, the error lies in $F^{N+1}\widehat{\mathcal{M}}_{\mathbb{L}}$. In particular,*

$$\frac{\{F_\sigma(n)\}}{\mathbb{L}^{10n+2-D(\sigma)}} \longrightarrow C(\sigma)\cdot \zeta_{\mathbb{P}^1\smallsetminus V_\sigma}\big(-\mathbb{L}^{-3}(\mathbb{L}-1)\big)$$

in $\widehat{\mathcal{M}}_{\mathbb{L}}$ as $N\to\infty$, for each $n \geq n_0(\sigma,N)$, where $\zeta_X(y) = \sum_{k\geq 0}\{\mathrm{Sym}^k(X)\}\, y^k$ is the Kapranov motivic zeta function.

Proof. Part (1) follows from the surjectivity condition (16): for $\ell \leq N$ and $n \geq n_0(\sigma,N)$, the jet evaluation is surjective and (17) applies.

For part (2), the error is $(-1)^{N+1}\{G_{\sigma,\geq N+1}(n)\}$, where $G_{\sigma,\geq N+1}(n)$ parametrizes coprime data satisfying the σ-conditions and having at least $N+1$ additional multiple roots of Δ on $\mathbb{P}^1 \smallsetminus V_\sigma$. Consider the incidence variety

$$\mathcal{I}_{N+1} \coloneqq \big\{\big((a_4,a_6),(w_1,\dots,w_{N+1})\big) \in G_\sigma(n)\times \mathrm{Conf}_{N+1}(\mathbb{P}^1\smallsetminus V_\sigma) : \mathrm{ord}_{w_i}(\Delta)\geq 2 \;\; \forall\, i\big\}.$$

Since $G_{\sigma,\geq N+1}(n)$ is the image of $\mathcal{I}_{N+1}$ under the first projection, $\dim G_{\sigma,\geq N+1} \leq \dim \mathcal{I}_{N+1}$. We estimate $\dim \mathcal{I}_{N+1}$ via the second projection $\pi_2 : \mathcal{I}_{N+1} \to \mathrm{Conf}_{N+1}(\mathbb{P}^1\smallsetminus V_\sigma)$.

Fix a generic configuration $w = (w_1,\dots,w_{N+1})$ with w_i pairwise distinct and $w_i \notin V_\sigma$. The fiber $\pi_2^{-1}(w)$ consists of $(a_4,a_6) \in G_\sigma(n)$ satisfying $\Delta(w_i) = 0$ and $\Delta'(w_i) = 0$ at each w_i. At w_i, the condition $\Delta(w_i) = 4a_4(w_i)^3 + 27a_6(w_i)^2 = 0$ is a single polynomial equation in the linear forms $a_4(w_i)$, $a_6(w_i)$ on the source. Since $4X^3+27Y^2$ is irreducible and nonconstant on $\{(X,Y)\neq(0,0)\}$, this condition cuts codimension 1 on the coprime locus. For generic w_i, the evaluation functionals $a_4(w_i)$, $a_6(w_i)$ are independent of the evaluation functionals at w_j $(j\neq i)$ and of the σ-jet constraints, since distinct-point evaluations on $H^0(\mathcal{O}(d))$ are linearly independent whenever the number of evaluation points does not exceed $d+1$, which holds for $n \geq n_0(\sigma,N)$. Hence $\Delta(w_i) = 0$ imposes a fresh codimension-1 condition at each w_i. Given $\Delta(w_i) = 0$, the derivative condition $\Delta'(w_i) = 12a_4(w_i)^2\, a_4'(w_i) + 54a_6(w_i)\, a_6'(w_i) = 0$ is a nontrivial linear equation in $a_4'(w_i)$, $a_6'(w_i)$ (by Lemma 2.10, since $(a_4(w_i),a_6(w_i)) \neq (0,0)$ on the coprime locus); for generic w_i this derivative evaluation is independent of the previous constraints by the same interpolation argument (the derivative evaluations $a_4'(w_i)$,

$a_6'(w_i)$ are independent of those at w_j for $j \neq i$ provided the total number of evaluation points does not exceed $\deg(a_4)+1 = 4n+1$, which again holds for $n \geq n_0(\sigma, N)$), cutting a further codimension 1. Therefore the $2(N+1)$ conditions at $w_1, \ldots, w_{N+1}$ are generically independent, giving

$$\dim \pi_2^{-1}(w) \;\leq\; (10n+2-D(\sigma))-2(N+1).$$

By the fiber dimension theorem [Har77, Ex. II.3.22], $\dim \mathcal{I}_{N+1} \leq \dim \pi_2^{-1}(w) + \dim \mathrm{Conf}_{N+1}(\mathbb{P}^1 \smallsetminus V_\sigma)$, so

$$\dim \mathcal{I}_{N+1} \;\leq\; (10n+2-D(\sigma)-2(N+1))+(N+1) \;=\; 10n+2-D(\sigma)-(N+1),$$

and hence $\dim G_{\sigma,\geq N+1} \leq 10n+2-D(\sigma)-(N+1)$, yielding (19). The bound holds vacuously if $G_{\sigma,\geq N+1}(n)$ is empty.

Part (3) follows: after normalizing by $\mathbb{L}^{10n+2-D(\sigma)}$, the error has dimension $\leq -(N+1)$, hence lies in $F^{N+1}\widehat{\mathcal{M}}_{\mathbb{L}}$ and converges to 0 as $N \to \infty$. The limit is the full inclusion–exclusion sum, which converges in $\widehat{\mathcal{M}}_{\mathbb{L}}$ to the stated expression. This adapts [VW15, Section 3] to the nonlinear discriminant $\Delta = 4a_4^3 + 27a_6^2$: the role of the growing line bundle in [VW15] is played by the growing height n, and the limit $\zeta_{\mathbb{P}^1 \smallsetminus V_\sigma}(-\alpha)$ with $\alpha = \mathbb{L}^{-3}(\mathbb{L}-1)$ is the motivic analog of the squarefree probability of [VW15, Theorem 1.13]. ∎

Theorem 2.12. *Let k be a perfect field with $\mathrm{char}(k) \neq 2, 3$. Set $K = k(t)$ and $s = t^{1/12}$. Then for each $m \geq 2$,*

$$[u^m] Z_{\mathrm{Triv}}(u;t) \;\in\; \overline{\widehat{\mathcal{M}}_{\mathbb{L}}(s)} \;\subset\; \widehat{\mathcal{M}}_{\mathbb{L}}[\![s]\!],$$

where $\widehat{\mathcal{M}}_{\mathbb{L}}$ is the completion of $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$ with respect to the dimensional filtration and $\overline{\widehat{\mathcal{M}}_{\mathbb{L}}(s)}$ denotes the closure of the rational functions $\widehat{\mathcal{M}}_{\mathbb{L}}(s)$ inside $\widehat{\mathcal{M}}_{\mathbb{L}}[\![s]\!]$ with respect to the coefficient-wise dimensional filtration topology. More precisely, for each $m \geq 2$ and each truncation depth $N \geq 0$, there exists a rational function $R_{m,N}(s) \in \widehat{\mathcal{M}}_{\mathbb{L}}(s)$ such that the coefficient of s^{12n} in $[u^m]Z_{\mathrm{Triv}}(u;t) - R_{m,N}(s)$ has dimension $\leq 10n-(N+1)$ for all $n \geq n_0(m,N)$. Setting $u=1$ recovers the exact Euler product of [BPS22, Thm. 8.9]*: $Z_{\mathrm{Triv}}(1;t) \in K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}](s)$.*

Proof. Work in the dimensional completion $\widehat{\mathcal{M}}_{\mathbb{L}}$ of $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}]$.

For $n=0$ the discriminant degree is 0, so the corresponding elliptic curve over $K=k(t)$ has everywhere good reduction and is constant. The moduli stack $\mathcal{W}_0^{\min}$ identifies with $\mathcal{M}_{1,1}$, so $\{\mathcal{W}_0^{\min}\} = \mathbb{L}$ by [Eke25] and the height-zero term is $u^2 \cdot \mathbb{L}$.

As in Proposition 2.9, decompose $Z_{\mathrm{Triv}} = Z_{\mathrm{Triv}}^{\circ} + Z_{\mathrm{Triv}}^{j=0} + Z_{\mathrm{Triv}}^{j=1728}$. By Proposition 2.8, the isotrivial Euler products $Z_{\mathrm{Triv}}^{j=0}$ and $Z_{\mathrm{Triv}}^{j=1728}$ are finite products of Kapranov factors, hence rational in s already in $K_0(\mathrm{Stck}_k)[\mathbb{L}^{-1}](s)$.

Fix $m \geq 2$ and extract $[u^m]Z_{\mathrm{Triv}}^{\circ}$. On $\mathcal{W}_n^{\circ}$ both $a_4 \not\equiv 0$ and $a_6 \not\equiv 0$, and each Weierstrass datum $(a_4, a_6) \in U_n^{\circ}$ determines a fiber configuration with $T(S) = 2 + \sum_v (m_v - 1)$. The visible fibers (those with $m_v - 1 \geq 1$) have total weight $\sum(m_v - 1) = m-2$, and the invisible fibers (II and I_1, each with $m_v - 1 = 0$) contribute

nothing to u^m. Let $\Sigma(m)$ be the finite set of visible configurations σ with $|\sigma|_{\mathrm{wt}} = m-2$. Then

$$[u^m]Z^{\circ}_{\mathrm{Triv}} = \sum_{\sigma\in\Sigma(m)}\sum_{n\geq 1}\frac{\{F_\sigma(n)\}}{\{\mathbb{G}_m\}\cdot\{\mathrm{PGL}_2\}}\,s^{12n}, \tag{20}$$

where $F_\sigma(n)$ is the locus of data in U_n° with visible configuration exactly σ at unordered support points on $\mathbb{P}^1$, $\mathbb{G}_m$ acts on (a_4,a_6) with weights $(4,6)$, and $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$ acts by reparametrization of the base.

For each $\sigma\in\Sigma(m)$, the additive visible fibers impose linear vanishing conditions at the support points with bounded total codimension $D(\sigma)\leq D(m)$ (cf. (14)). On the non-constant-j locus, type-II fibers satisfy $\nu_v(a_4)\geq 1$; since $a_4\not\equiv 0$ has degree $\leq 4n$, the total type-II multiplicity is at most $4n$. Since II is invisible, these do not affect the additive visible codimension. By the sparsity argument of Lemma 2.7, the additive conditions are independent for $n\geq n_1(\sigma)$. The multiplicative visible fibers (I_{k_i} with $k_i\geq 2$ and the cusp components of $\mathrm{I}^*_{k_j}$) have total jet demand bounded by $D(m)$, so by Lemma 2.6 the jet evaluation map at the visible support points is surjective for $n\geq n_0(\sigma) := \max\big(\lceil (D_{a_4}(\sigma)-1)/4\rceil, \lceil (D_{a_6}(\sigma)-1)/6\rceil\big)$.

For $n\geq \max(n_0(\sigma),n_1(\sigma))$, the class $\{G_\sigma(n)\}$ satisfies (15): $\{G_\sigma(n)\} = C(\sigma)\cdot \mathbb{L}^{10n+2-D(\sigma)} + E_\sigma(n)$, where $G_\sigma(n)\supseteq F_\sigma(n)$ includes data with possible extra visible fibers, $C(\sigma)$ depends on σ alone, and $\dim(E_\sigma(n))\leq 6n+1-D_{a_6}(\sigma)$. To pass from $G_\sigma(n)$ to $F_\sigma(n)$, we remove the extra visible fibers at points outside the σ-support. By Proposition 2.11, for each truncation depth N and $n\geq n_0(\sigma,N)$, the truncated approximation $F_\sigma^{(N)}(n)$ satisfies (17) and the error $F_\sigma(n)-F_\sigma^{(N)}(n)$ has dimension $\leq 10n+2-D(\sigma)-(N+1)$. The generating function $\sum_{n\geq n_0}F_\sigma^{(N)}(n)s^{12n}$ is rational in s: each summand in (18) is a product of a motivic constant, a power of $\mathbb{L}^n$, and a symmetric power class, hence contributes a rational function of s. The error series has coefficients of dimension $\leq 10n+2-D(\sigma)-(N+1)$, so for each N the generating function $\sum_n\{F_\sigma(n)\}s^{12n}$ is within dimensional distance $N+1$ of a rational function of s. Since N is arbitrary, $\sum_n\{F_\sigma(n)\}s^{12n}\in\overline{\widehat{\mathcal{M}}_{\mathbb{L}}(s)}$.

For $n<\max(n_0(\sigma,N),n_1(\sigma))$ (finitely many heights depending on m), the contributions $\sum_{n<n_0}\{F_\sigma(n)\}s^{12n}$ form a polynomial in s, which is rational. Summing over the finite set $\Sigma(m)$ and dividing by $\{\mathbb{G}_m\}\cdot\{\mathrm{PGL}_2\}$, we conclude $[u^m]Z^\circ_{\mathrm{Triv}}\in\overline{\widehat{\mathcal{M}}_{\mathbb{L}}(s)}$. Combined with the isotrivial Euler products and the height-zero term, $[u^m]Z_{\mathrm{Triv}}\in\overline{\widehat{\mathcal{M}}_{\mathbb{L}}(s)}$ for each m.

At $u=1$ the trivial lattice grading is forgotten and $Z_{\mathrm{Triv}}(1;t) = Z_{\vec{1}}(t)$, the unweighted motivic height zeta function. By [BPS22, Thm. 8.9], this equals an exact Euler product: the stabilization of Proposition 2.11 is exact at $u=1$ because every visible configuration contributes with weight 1, the overcounting from G_σ cancels in the sum over all σ, and the correction vanishes. ■

As a consequence of Theorem 2.12 and the weight bound in mixed Hodge theory, over $k=\mathbb{C}$ the approximate motivic rationality yields bidegree-wise Hodge number stabilization.

Corollary 2.13. *Let $k = \mathbb{C}$, $K = \mathbb{C}(t)$, and $s = t^{1/12}$. For each $m \geq 2$ and each bidegree $(p_0, q_0) \in \mathbb{Z}^2$, the sequence $n \mapsto e_c^{p_0,q_0}\big(\{\mathcal{W}_n^{\min}(m)\} \cdot \mathbb{L}^{-10n}\big)$ is eventually constant, and the generating function $\sum_{n\geq 0} e_c^{p_0,q_0}\big(\{\mathcal{W}_n^{\min}(m)\} \cdot \mathbb{L}^{-10n}\big)s^{12n}$ is a rational function of s.*

Proof. The weight bound in mixed Hodge theory states: for $\alpha \in K_0(\mathrm{Stck}_{\mathbb{C}})[\mathbb{L}^{-1}]$ with $\dim(\alpha) \leq d$, one has $e^{p,q}(\alpha) = 0$ for $p + q > 2d$. This holds by reduction to the variety case via the scissor relations [Eke25, Thm. 1.2] and Deligne's weight bound [Del74].

Fix $m \geq 2$ and a visible configuration $\sigma \in \Sigma(m)$. By Proposition 2.11, for each truncation depth $N \geq 0$ and $n \geq n_0(\sigma, N)$, the normalized truncated approximation $F_\sigma^{(N)}(n) \cdot \mathbb{L}^{-(10n+2-D(\sigma))}$ equals a motivic class $K_{\sigma,N} \in \mathcal{M}_{\mathbb{L}}$ independent of n: the n-dependence in (17) enters only through the kernel factor $\mathbb{L}^{10n+2-D(\sigma)-4\ell}$, which is absorbed by the normalization. The normalized truncation error $(F_\sigma(n) - F_\sigma^{(N)}(n)) \cdot \mathbb{L}^{-(10n+2-D(\sigma))}$ has dimension $\leq -(N+1)$ by Proposition 2.11(2), and the isotrivial correction $E_\sigma(n)$ of (15) has normalized dimension $\leq -(4n + 1 - D_{a_4}(\sigma))$. By the weight bound, both contributions vanish at any fixed bidegree (p_0, q_0) once $N \geq \lceil -(p_0 + q_0)/2 \rceil$ and n is sufficiently large.

Therefore, for each (p_0, q_0), the Hodge number $e^{p_0,q_0}(F_\sigma(n) \cdot \mathbb{L}^{-(10n+2-D(\sigma))})$ equals $[x^{p_0} y^{q_0}] K_{\sigma,N}$ for all $n \geq n_0(m, N)$, and this value stabilizes for $N \geq \lceil -(p_0 + q_0)/2 \rceil$. The full coefficient $\{\mathcal{W}_n^{\min}(m)\} \cdot \mathbb{L}^{-10n}$ is assembled from the finite sum over $\sigma \in \Sigma(m)$, the isotrivial Euler products (which are strictly rational in s by Proposition 2.8, hence contribute eventually constant Hodge numbers), and finitely many small-height corrections (a polynomial in s). Each Hodge number $e_c^{p_0,q_0}(\{\mathcal{W}_n^{\min}(m)\} \cdot \mathbb{L}^{-10n})$ is therefore eventually constant in n, and an eventually constant sequence satisfies a linear recurrence, giving rationality in s. ∎

The explicit Euler product $E(u; s)$ built from Table 1 agrees with $Z_{\mathrm{Triv}}(u; t)$ at $u = 1$ by [BPS22, Thm. 8.9] and to arbitrary dimensional precision for general u by Theorem 2.12. We conjecture exact agreement:

Conjecture 2.14. For each reduction type Θ in Table 1, let $y_\Theta(u; s)$ denote the argument of $\zeta_{\mathbb{P}^1}$ in the displayed Euler factor, and set

$$E(u; s) \coloneqq u^2 \cdot \mathbb{L} \cdot \prod_{\Theta} \zeta_{\mathbb{P}^1}\big(y_\Theta(u; s)\big). \tag{21}$$

Then $[u^m] Z_{\mathrm{Triv}}(u; t) = [u^m] E(u; s)$ in $\widehat{\mathcal{M}_{\mathbb{L}}}(s)$ for each $m \geq 2$.

Since $E(u; s)$ is rational in both u and s, the conjecture would give strict bivariate rationality of $Z_{\mathrm{Triv}}(u; t)$; it is possible, however, that the coefficient-wise approximate rationality of Theorem 2.12 is optimal, since the stabilization threshold $n_0(m, N)$ grows with m and no uniform bound is available.

Remark 2.15. Assume $\mathrm{char}(k) \neq 2, 3$. The local factor coefficients A_Θ in Table 1 arise from two distinct mechanisms (cf. [BPS22, Thm. 1.6])

Additive types. For each additive Kodaira type $\Theta \in \{\mathrm{II}, \mathrm{III}, \mathrm{IV}, \mathrm{I}_0^*(j \neq 0, 1728), \mathrm{I}_0^*(j \in \{0, 1728\}), \mathrm{IV}^*, \mathrm{III}^*, \mathrm{II}^*\}$, the coefficient A_Θ is the normalized one-fiber motivic class of the stratum $\mathcal{W}_{n,\mathbb{P}^1}^{\Theta}$ parametrizing minimal fibrations with exactly one singular

fiber of type Θ and semistable fibers elsewhere. After the quotient by $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2$[1] and the height-dependent factor $\mathbb{L}^{10n-20}$:

$$A_\Theta := \frac{\{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}}{\{\mathrm{PGL}_2\}\,\mathbb{L}^{10n-20}}.$$

This normalization differs from that in the proof of Lemma 2.7 ($A_j = \{\mathcal{W}^{\Theta}_{n,\mathbb{P}^1}\}/(\{\mathbb{P}^1\}\cdot \mathbb{L}^{10n})$) by a universal factor independent of Θ, absorbed into the prefactor.

Cusp families. For the multiplicative family I_k ($k \geq 1$), no additive vanishing is imposed on (a_4, a_6): the contact order $k = \mathrm{ord}_v(\Delta)$ is determined entirely by the nonlinear discriminant $\Delta = 4a_4^3 + 27a_6^2$. The coefficient $A_{\mathrm{I}_k} = \mathbb{L}^{18}$ for all $k \geq 1$ is the normalized motivic class of the local multiplicative locus $\Lambda(k)$ of Definition 2.5 (originally [HP19, Cor. 2]). For the additive cusp family I^*_k ($k \geq 1$), the local condition imposes additive vanishing $(p, q) = (2, 3)$ (the same as $\mathrm{I}^*_0(j \neq 0, 1728)$; cf. [BPS22, Thm. 1.6]) together with a cusp contact order k determined by the discriminant; the normalized class is $A_{\mathrm{I}^*_k} = \mathbb{L}^{14} - \mathbb{L}^{13}$ for all $k \geq 1$. In both cases, the k-independence is what allows the cusp resummation (Lemma 2.4) to collapse the k-dependence into $(1-us)^{-1}$.

Then [BPS22, Thm. 1.8] determines the following one-fiber motivic classes.

Conventions. In Table 1, $\delta(s) := 1-us$ denotes the cusp resummation denominator of Lemma 2.4; this is not to be confused with the Weierstrass discriminant $\Delta = 4a_4^3 + 27a_6^2$. For each reduction type Θ, let $y_\Theta(u; s)$ denote the local monomial in the displayed denominator; we record only the reduced factor $(1-y_\Theta)^{-1}$, the full $\mathbb{P}^1$-contribution being

$$(1-y_\Theta)^{-\{\mathbb{P}^1\}} = \frac{1}{\big(1-y_\Theta\big)\big(1-\mathbb{L}\,y_\Theta\big)}.$$

3. Kudla–Millson modularity and the canonical height distribution

The approximate rationality of $Z_{\mathrm{Triv}}(u; t)$ rests on the locality of the trivial lattice rank: $T(S)$ depends only on the fiber configuration $\mathfrak{f}$, hence is constant on each Kodaira stratum, and unordered collections of local factors assemble into a finite Euler product. In contrast, the Mordell–Weil rank $\mathrm{rk}(E/K)$ is not determined by the fiber configuration. On a fixed Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$ with $T(S) = T(\mathfrak{f})$ constant, the Shioda–Tate formula shows that variation of $\mathrm{rk}(E/K)$ is equivalent to variation of $\rho(S)$. Inside such a stratum, imposing that $\mathrm{NS}(S_{\bar{k}})$ contain additional algebraic classes independent of the trivial lattice defines Noether–Lefschetz loci, which by the theorem of Cattani–Deligne–Kaplan [CDK95] form a countable union of closed algebraic subsets that is not constructible. The local-to-global factorization producing a finite Euler product for Z_{Triv} structurally fails for Z_{MW} and Z_{NS}.

The moduli stack $\mathcal{W}_n^{\min,(\mathfrak{f})}$ in Proposition 2.1 provides the geometric input that converts this obstruction into an arithmetic statement: modularity of special cycles on period domains yields transcendence of the distribution of new algebraic

[1]The *unparameterized* $\mathbb{P}^1_k$ corresponds to taking the PGL_2 stack quotient; motivically this factors out $\{\mathrm{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$. See [PS25] for a comprehensive treatment.

Reduction type Θ	$\gamma : (v(a_4),\ v(a_6))$	(r,a)	$m_v - 1$	$\mathbb{P}^1$-Euler factor in $Z_{\mathrm{Triv}}(u;t)$
$\mathrm{I}_{k\geq 1}$ $(j=\infty)$	$(0,0)$	$\emptyset$	$k-1$	$\dfrac{1}{1-\mathbb{L}^{18}\,s\,\delta(s)^{-1}}$
II $(j=0)$	$(\geq 1,\ 1)$	$(6,1)$	0	$\dfrac{1}{1-\mathbb{L}^{17}\,s^2}$
III $(j=1728)$	$(1,\ \geq 2)$	$(4,1)$	1	$\dfrac{1}{1-\mathbb{L}^{16}\,u\,s^3}$
IV $(j=0)$	$(\geq 2,\ 2)$	$(3,1)$	2	$\dfrac{1}{1-\mathbb{L}^{15}\,u^2\,s^4}$
$\mathrm{I}^*_{k\geq 1}$ $(j=\infty)$	$(2,\ 3)$	$(2,1)$	$k+4$	$\dfrac{1}{1-(\mathbb{L}^{14}-\mathbb{L}^{13})\,u^5\,s^7\,\delta(s)^{-1}}$
I^*_0 $(j\neq 0,1728)$	$(2,\ 3)$	$(2,1)$	4	$\dfrac{1}{1-(\mathbb{L}^{14}-\mathbb{L}^{13})\,u^4\,s^6}$
I^*_0 $(j=0)$	$(\geq 3,\ 3)$	$(2,1)$	4	$\dfrac{1}{1-\mathbb{L}^{13}\,u^4\,s^6}$
I^*_0 $(j=1728)$	$(2,\ \geq 4)$	$(2,1)$	4	$\dfrac{1}{1-\mathbb{L}^{13}\,u^4\,s^6}$
IV^* $(j=0)$	$(\geq 3,\ 4)$	$(3,2)$	6	$\dfrac{1}{1-\mathbb{L}^{12}\,u^6\,s^8}$
III^* $(j=1728)$	$(3,\ \geq 5)$	$(4,3)$	7	$\dfrac{1}{1-\mathbb{L}^{11}\,u^7\,s^9}$
II^* $(j=0)$	$(\geq 4,\ 5)$	$(6,5)$	8	$\dfrac{1}{1-\mathbb{L}^{10}\,u^8\,s^{10}}$

TABLE 1. Local factors for $\mathcal{P}(4,6)$.

classes by canonical height d at every height $n \geq 2$. Over $k = \mathbb{C}$, on a fixed Kodaira stratum, any Noether–Lefschetz class appearing newly must have nonzero Mordell–Weil projection by the Shioda–Tate formula, forcing $\mathrm{rk}(E/K) \geq 1$.

3.1. **Period domains and special cycles at height n.** At height $n \geq 2$, the stable stratum $\mathcal{W}_n^{\mathrm{st}}(T{=}2)$ parametrizes elliptic surfaces with section and $12n$ nodal fibers of type I_1. More generally, the *irreducible-fibers locus* $\mathcal{W}_n^{\mathrm{irr}}(T{=}2) \supseteq \mathcal{W}_n^{\mathrm{st}}$ parametrizes surfaces whose singular fibers are all irreducible (types I_1 and II);

on this locus $R(\mathfrak{f}) = 0$ and the Heegner to Mordell–Weil implication holds for all $d \geq 1$. For any surface $[S] \in \mathcal{W}_n^{\min}$, the orthogonal complement of the sublattice $\langle F, O\rangle$ spanned by the fiber class and zero section

$$\Lambda_n \coloneqq \langle F, O\rangle^{\perp} \subset H^2(S, \mathbb{Z})$$

has rank $12n-4$ and signature $(2n-2,\, 10n-2)$. The lattice Λ_n is unimodular (since $\langle F, O\rangle$ is unimodular and $H^2(S, \mathbb{Z})$ is unimodular by Poincaré duality) and even (by the Wu formula: $\gamma^2 \equiv \gamma \cdot K_S = (n-2)\,\gamma \cdot F = 0 \pmod 2$ for all $\gamma \in \langle F, O\rangle^{\perp}$). By the classification of indefinite even unimodular lattices [Ser73, Ch. V, Thm. 5],

$$\Lambda_n \cong U^{\oplus(2n-2)} \oplus E_8(-1)^{\oplus n},$$

where U denotes the rank-2 hyperbolic lattice. The classifying space for Hodge structures on Λ_n is

$$\mathcal{D}_n \simeq \mathrm{SO}(2n{-}2,\, 10n{-}2)\big/U(n{-}1) \times \mathrm{SO}(10n{-}2),$$

which is a Hermitian symmetric domain (type IV) when $n = 2$ and a Griffiths domain when $n \geq 3$. For each integer $d \geq 1$, the Heegner locus

$$\mathcal{H}_d \coloneqq \left\{x \in \mathcal{D}_n : \exists\, \gamma \in \Lambda_n,\, \gamma^2 = -2d,\, \gamma \perp H^{2,0}_x\right\}$$

is a locally finite union of codimension-$(n-1)$ complex submanifolds of $\mathcal{D}_n$. By Borel's finiteness theorem [Bor69], $O(\Lambda_n, \mathbb{Z})$ acts with finitely many orbits on norm $-2d$ lattice vectors, so $\mathcal{H}_d$ descends to a finite union of analytic subvarieties of codimension $n-1$ on the arithmetic quotient $O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n$, defining a cohomology class $[\mathcal{H}_d] \in H^{2(n-1)}(O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n,\, \mathbb{Q})$. Via the period map, the pullback $\Phi^{-1}(\mathcal{H}_d) \subset \mathcal{W}_n^{\mathrm{st}}$ is a closed algebraic subvariety by [CDK95].

3.2. **Modularity: Borcherds for $n = 2$, Kudla–Millson for all n.** At $n = 2$, Λ_2 has signature $(2, 18)$ and $\mathcal{D}_2$ is a type IV Hermitian symmetric domain. The Borcherds theorem [Bor99] gives modularity in the Chow group:

Theorem 3.1 (Borcherds [Bor99, Thm. 4.5])**.** *Let L be an even unimodular lattice of signature $(2, b^-)$ with $b^- \geq 3$, and let $\Gamma \subset O(L)$ be a subgroup of finite index. The formal generating series $\sum_{d\geq 0}[\mathcal{H}_d]\,q^d$ of Heegner divisor classes is a modular form of weight $1 + b^-/2$ for $\mathrm{SL}_2(\mathbb{Z})$, valued in $\mathrm{CH}^1(\Gamma\backslash\mathcal{D})$.*

At $n = 2$ with $b^- = 18$, this gives a modular form of weight 10 in the Chow group; the resulting Noether–Lefschetz numbers for K3 surfaces were computed explicitly by Maulik–Pandharipande [MP13]. However, Borcherds requires signature $(2, b^-)$ and does not extend to $n \geq 3$.

Although the period domain $\mathcal{D}_n$ is a Griffiths domain for $n \geq 3$, and not Hermitian symmetric, Greer [Gre20, Thm. 37] established modularity by pulling back the Kudla–Millson theorem from the underlying symmetric space M_n to $\mathcal{D}_n$. We will use the following formulation due to Garcia, which applies directly to the Griffiths domain $\mathcal{D}_n$ without requiring it to be Hermitian symmetric.

Theorem 3.2 (Garcia [Gar18, Thm. 1.2])**.** *Let $(V_{\mathbb{Q}}, Q)$ be a rational quadratic space, $L \subset V_{\mathbb{Q}}$ an even integral lattice with dual lattice $L^{\vee} \supseteq L$, and $\mathcal{D}$ the period domain parametrizing polarized Hodge structures of even weight on $V_{\mathbb{Q}}$. Let $\Gamma_L \coloneqq \{\gamma \in$*

$O^+(V_{\mathbb{R}}, Q) \mid \gamma(L) = L,\ \gamma|_{L^\vee/L} = \mathrm{id}\}$. *Then the generating series of Hodge loci*

$$c_{\mathrm{top}}(\mathcal{F}^\vee) \;+\; \sum_{d\geq 1} \mathrm{Hdg}(d, L)\, q^d$$

is a holomorphic modular form of weight $\frac{1}{2}\dim V$*, valued in* $H^{2\,\mathrm{rk}(\mathcal{F})}(\Gamma_L\backslash\mathcal{D}, \mathbb{Q})$*. When* L *is unimodular,* $L^\vee = L$ *and the modular form is scalar-valued for* $\mathrm{SL}_2(\mathbb{Z})$*; for general* L *it transforms under the Weil representation of the discriminant group* $L^\vee/L$*.*

Applied to Λ_n with $\dim \Lambda_n = 12n-4$ and Hodge bundle $\mathcal{F} = H^{2,0}$ of rank $n-1$:

Corollary 3.3. *For each* $n \geq 2$*, the generating series of Noether–Lefschetz classes on the arithmetic quotient* $O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n$ *is a cohomology-valued modular form of weight*

$$\tfrac{1}{2}(12n-4) \;=\; 6n-2$$

for $\mathrm{SL}_2(\mathbb{Z})$*, valued in* $H^{2(n-1)}(O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n, \mathbb{Q})$*. The scalar-valuedness and* $\mathrm{SL}_2(\mathbb{Z})$*-modularity hold because* Λ_n *is unimodular. At* $n = 2$*, this recovers the Kudla–Millson theorem* [KM90] *on the type IV domain, and Borcherds* [Bor99] *refines the statement to* CH^1*.*

The period map is $\Phi\colon \mathcal{W}_n^{\mathrm{st}} \to O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n$. We work on this arithmetic quotient throughout. For the transcendence conclusion below, we need the generating series to be a nonzero modular form, which requires the Heegner classes $[\mathcal{H}_d]$ to be nonzero in cohomology.

Lemma 3.4. *For each* $n \geq 2$ *and all* $d \gg 0$*, the Heegner class* $[\mathcal{H}_d]$ *is nonzero in* $H^{2(n-1)}(O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n, \mathbb{Q})$*.*

Proof. Let $\Gamma' \subset O(\Lambda_n, \mathbb{Z})$ be a torsion-free finite-index normal subgroup, which exists by Selberg's lemma. The period domain fibres over the symmetric space,

$$g\colon \Gamma'\backslash\mathcal{D}_n \;\longrightarrow\; \Gamma'\backslash M_n, \qquad M_n := \mathrm{SO}(2n{-}2,\, 10n{-}2)\big/\,\mathrm{SO}(2n{-}2)\times\mathrm{SO}(10n{-}2),$$

with fiber $Y = \mathrm{SO}(2n{-}2)/U(n{-}1)$, and the Heegner loci are g-preimages of the Kudla–Millson special cycles: $\mathcal{H}_d = g^{-1}(\tilde{C}_{2d})$ [Gre20, §5]. For every $d \geq 1$, the lattice Λ_n contains a vector of norm $-2d$ (since Λ_n contains a copy of the hyperbolic plane $U = \langle e, f\rangle$ with $e^2 = f^2 = 0$, $e\cdot f = 1$, and $v = e - df$ has $v^2 = -2d$), so $\tilde{C}_{2d}$ is nonempty.

To see that it represents a nonzero cohomology class, we use Kudla–Millson modularity [KM90, Thm. 2]: the generating series $\sum_{d\geq 0}[\tilde{C}_{2d}]q^d$ is a modular form of weight $6n-2$ for $\mathrm{SL}_2(\mathbb{Z})$ valued in $H^{2n-2}(\Gamma'\backslash M_n, \mathbb{Q})$. Its constant term $[\tilde{C}_0] = e(\mathcal{V}^\vee)$, the Euler class of the dual tautological bundle, is nonzero [Gre20, Thm. 37]. Because the space of Eisenstein series of weight $6n{-}2$ for $\mathrm{SL}_2(\mathbb{Z})$ is one-dimensional, the Eisenstein component of this modular form is exactly $E_{6n-2}(q) \otimes c\, e(\mathcal{V}^\vee)$ for some nonzero constant c. Thus, its d-th Fourier coefficient contributes a main term that is a nonzero multiple of $\sigma_{6n-3}(d)\cdot e(\mathcal{V}^\vee)$. By Deligne's bound, the cuspidal Fourier coefficients are $O(d^{(6n-3)/2+\epsilon})$. Since $n \geq 2$, this cuspidal error is asymptotically dominated by the Eisenstein main term, which grows at least as fast as d^{6n-3}. Hence $[\tilde{C}_{2d}] \neq 0$ in $H^{2n-2}(\Gamma'\backslash M_n, \mathbb{Q})$ for all $d \gg 0$.

The cohomology of the fiber $Y = \mathrm{SO}(2n-2)/U(n-1)$ is generated by Chern classes of the tautological complex bundle [Bor53], which extend to the total space as Chern classes of the Hodge bundle $\mathcal{F}$. By Leray–Hirsch, $g^*\colon H^p(\Gamma'\backslash M_n, \mathbb{Q}) \hookrightarrow$

$H^p(\Gamma'\backslash\mathcal{D}_n,\mathbb{Q})$ is injective, so $[\mathcal{H}_d]=g^*[\tilde{C}_{2d}]\neq 0$. Finally, the projection $\Gamma'\backslash\mathcal{D}_n\to O(\Lambda_n,\mathbb{Z})\backslash\mathcal{D}_n$ is a finite cover, so pullback is injective on rational cohomology, and the class $[\mathcal{H}_d]$ remains nonzero on $O(\Lambda_n,\mathbb{Z})\backslash\mathcal{D}_n$. ∎

The modularity of the Noether–Lefschetz generating series for elliptic surfaces over $\mathbb{P}^1$, in all Siegel genera, was established by Greer [Gre20, Thm. 37] by pulling back the Kudla–Millson theorem [KM90] on the symmetric space M_n to the period domain $\mathcal{D}_n$ via the natural fibration; Garcia [Gar18, Thm. 1.2] gives an alternative proof via Quillen superconnections, working intrinsically on $\mathcal{D}_n$. In both treatments, the generating series is a scalar-valued modular form for $\mathrm{SL}_2(\mathbb{Z})$ because the full primitive lattice $\Lambda_n=\langle F,O\rangle^\perp$ is even unimodular of signature $(2n{-}2,\,10n{-}2)$ on all of $\mathcal{W}_n^{\min}$.

To isolate the Mordell–Weil contribution from root-lattice contributions (i.e., algebraic classes from reducible singular fibers) requires a finer stratification: on each Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$ the root lattice $R(\mathfrak{f})\subset\Lambda_n$ is constant and algebraic throughout the stratum, so a newly algebraic class $\gamma\in\Lambda_n$ on the stratum cannot lie in $R(\mathfrak{f})$ and therefore projects non-trivially to $\mathrm{MW}(E/K)$ under Shioda–Tate. This per-Kodaira-stratum argument relies on the constructible stratification of $\mathcal{W}_n^{\min}$ by Kodaira type established in [BPS22, Thms. 5.1 and 7.12].

Throughout this section, $\sigma(P)\in\Lambda_n\otimes\mathbb{Q}$ denotes the image of $P\in E(K)$ under the Shioda map [Shi90, §8], and $\hat{h}(P):=-\frac{1}{2}\sigma(P)^2$ the normalized canonical height, so that $\gamma^2=-2d$ corresponds to $\hat{h}(P)=d$.

Theorem 3.5. *For each $n\geq 2$, let $\Lambda_n=\langle F,O\rangle^\perp\subset H^2(S,\mathbb{Z})$ be the orthogonal complement of the sublattice spanned by the fiber class and zero section, an even unimodular lattice of signature $(2n{-}2,\,10n{-}2)$ and rank $12n{-}4$. Let $\mathcal{D}_n$ be the associated period domain.*

(1) *The generating series of Noether–Lefschetz classes*

$$c_{\mathrm{top}}(\mathcal{F}^\vee)\;+\;\sum_{d\geq 1}[\mathcal{H}_d]\,q^d\;\in\;H^{2(n-1)}(O(\Lambda_n,\mathbb{Z})\backslash\mathcal{D}_n,\,\mathbb{Q})[\![q]\!]$$

is a holomorphic modular form of weight $6n-2$ for $\mathrm{SL}_2(\mathbb{Z})$, valued in $H^{2(n-1)}(O(\Lambda_n,\mathbb{Z})\backslash\mathcal{D}_n,\,\mathbb{Q})$. The Fourier coefficient $[\mathcal{H}_d]$ is the cycle class of the Heegner locus parametrizing Hodge structures where a lattice vector $\gamma\in\Lambda_n$ with $\gamma^2=-2d$ becomes algebraic.

(2) *On any fixed Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$, the root lattice $R(\mathfrak{f})\subset\Lambda_n$ is constant and algebraic throughout. A class $\gamma\in\Lambda_n$ with $\gamma^2=-2d$ that becomes newly algebraic on the stratum cannot lie in $R(\mathfrak{f})$, hence forces $\mathrm{rk}(E/K)\geq 1$ by the Shioda–Tate exact sequence. The corresponding section has canonical height $\hat{h}(P)\leq d$, with equality when all singular fibers are irreducible (types I_1 and II). The genus-r Siegel theta series restricted to $R(\mathfrak{f})^\perp\cap\Lambda_n$ detects $\mathrm{rk}(E/K)\geq r$ for each $1\leq r\leq 10n-T(\mathfrak{f})$.*

(3) *For any $\alpha\in H_{2(n-1)}(O(\Lambda_n,\mathbb{Z})\backslash\mathcal{D}_n,\,\mathbb{Q})$ such that $\alpha\cap[\mathcal{H}_d]\neq 0$ for some d, the scalar-valued series*

$$\varphi_{n,\alpha}(q)\;=\;\alpha\cap c_{\mathrm{top}}(\mathcal{F}^\vee)\;+\;\sum_{d\geq 1}(\alpha\cap[\mathcal{H}_d])\,q^d\;\in\;M_{6n-2}(\mathrm{SL}_2(\mathbb{Z}))$$

is transcendental over $\mathbb{C}(q)$. Such α exist by Lemma 3.4.

Proof. For (1), this is Corollary 3.3 applied to the unimodular lattice Λ_n.

For (2), the key input is the constructible stratification of Proposition 2.1: at each height n, the moduli stack $\mathcal{W}_n^{\min}$ decomposes into finitely many Kodaira strata $\mathcal{W}_n^{\min,(\mathfrak{f})}$, each a locally closed substack on which the fiber configuration $\mathfrak{f}$ is constant. The root lattice $R(\mathfrak{f}) \subset \Lambda_n$, spanned by the irreducible fiber components not meeting the zero section, is therefore constant on the stratum, and its classes are of type $(1,1)$ at every point.

Suppose $\gamma \in \Lambda_n$ with $\gamma^2 = -2d$ becomes of type $(1,1)$ at $[S] \in \mathcal{W}_n^{\min,(\mathfrak{f})}$. If $\gamma \in R(\mathfrak{f})$, then γ is already of type $(1,1)$ everywhere on the stratum, contradicting the assumption that γ is newly algebraic. Hence $\gamma \notin R(\mathfrak{f})$. Since $\gamma \in \Lambda_n$ is also orthogonal to $\langle F, O\rangle$, it does not lie in $\mathrm{Triv}(S) = \langle F, O\rangle \oplus R(\mathfrak{f})$. The Shioda–Tate exact sequence $0 \to \mathrm{Triv}(S) \to \mathrm{NS}(S_{\bar{k}}) \to \mathrm{MW}(E/K) \to 0$ therefore shows that γ maps nontrivially to $\mathrm{MW}(E/K)$, giving $\mathrm{rk}(E/K) \geq 1$ (cf. [Shi90, §5, Thm. 1.3]).

To bound the canonical height, let $\gamma_R \in R(\mathfrak{f}) \otimes \mathbb{Q}$ and $\gamma_{\mathrm{MW}} \in (R(\mathfrak{f})^{\perp} \cap \Lambda_n) \otimes \mathbb{Q}$ be the orthogonal projections of γ in $\Lambda_n \otimes \mathbb{Q}$ [Shi90, §8, Lem. 8.1]. Since $R(\mathfrak{f})$ is negative definite, $\gamma_R^2 \leq 0$, and the image of γ in $\mathrm{MW}(E/K) \otimes \mathbb{Q}$ under the Shioda map has canonical height

$$\hat{h}(P) \;=\; -\gamma_{\mathrm{MW}}^2/2 \;=\; d + \gamma_R^2/2 \;\leq\; d,$$

where equality holds if and only if $\gamma_R = 0$, which is guaranteed, for instance, when all singular fibers are irreducible (types I_1 and II).

The higher-rank statement follows by applying the same argument to r independent classes $\gamma_1, \ldots, \gamma_r$ simultaneously. The genus-r Siegel series is Garcia [Gar18, Thm. 1.2] applied to the restricted lattice $R(\mathfrak{f})^{\perp} \cap \Lambda_n$ at genus r. Nonemptiness of the Hodge loci $\mathrm{Hdg}(T, R(\mathfrak{f})^{\perp} \cap \Lambda_n)$ for arbitrarily large positive definite T of rank $r \leq 10n - T(\mathfrak{f})$ is immediate: in Garcia's convention $Q = -(\text{intersection form})$, the Q-positive definite part of $(R(\mathfrak{f})^{\perp} \cap \Lambda_n) \otimes \mathbb{R}$ has dimension $10n - T(\mathfrak{f}) \geq r$, so $R(\mathfrak{f})^{\perp} \cap \Lambda_n$ contains rank-r Q-positive definite sublattices, and scaling produces forms of arbitrarily large determinant.

For (3), by Lemma 3.4, $[\mathcal{H}_d] \neq 0$ in $H^{2(n-1)}(O(\Lambda_n, \mathbb{Z})\backslash \mathcal{D}_n, \mathbb{Q})$ for $d \gg 0$. The universal coefficient theorem provides α with $\alpha \cap [\mathcal{H}_d] \neq 0$. Pairing the cohomology-valued series of part (1) with α gives a scalar-valued modular form $\varphi_{n,\alpha} \in M_{6n-2}(\mathrm{SL}_2(\mathbb{Z}))$, which is nonzero since $\alpha \cap [\mathcal{H}_d] \neq 0$. At $n = 2$, $\dim M_{10}(\mathrm{SL}_2(\mathbb{Z})) = 1$, so $\varphi_{2,\alpha} \propto E_{10}$. A nonzero modular form of positive weight is transcendental over $\mathbb{C}(q)$. ∎

The per-Kodaira-stratum hypothesis in Theorem 3.5(2) is necessary, since a stratum-independent statement does not hold in general. On a Kodaira stratum $\mathcal{W}_n^{\min,(\mathfrak{f})}$ with root lattice $R(\mathfrak{f}) \subset \Lambda_n$, every point of the stratum satisfies the Heegner condition at index d whenever $R(\mathfrak{f})$ represents d, i.e., whenever there exists $\gamma \in R(\mathfrak{f})$ with $\gamma^2 = -2d$. Such a class is algebraic throughout the stratum and has zero Mordell–Weil projection. For example, on a stratum with a single I_2 fiber and all remaining singular fibers of type I_1, $R(\mathfrak{f}) = A_1$ contains a root of norm -2, so the Heegner condition at $d = 1$ is satisfied throughout the stratum with no Mordell–Weil rank jump. On any stratum with all singular fibers irreducible (types I_1 and II), $R(\mathfrak{f}) = 0$ and the implication holds for all $d \geq 1$.

Although Noether–Lefschetz loci are defined via Hodge theory, their arithmetic content is visible over finite fields: the loci where $\rho(S)$ jumps are detected by

Frobenius eigenvalues on $H^2_{\text{ét}}(S_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell)$ acquiring additional algebraic coincidences (cf. [Tay25] for the analogous phenomenon for K3 surfaces over $\mathbb{F}_p$). However, the theta correspondence underlying Theorem 3.5 is a real-analytic construction with no known ℓ-adic analogue, and for $n \geq 3$ the period domain is not a Shimura variety, so the existing modularity frameworks do not extend directly to $\mathbb{F}_q(t)$.

Theorem 3.5 gives a modularity ladder indexed by Siegel genus: the Fourier coefficients of the genus-r theta lift on Λ_n detect elliptic surfaces with $\mathrm{rk}(E/K) \geq r$ in the period domain. However, non-emptiness of Hodge loci $\mathrm{Hdg}(T, \Lambda_n)$ in $\mathcal{D}_n$ does not by itself produce an elliptic surface with $\mathrm{rk}(E/K) \geq r$: for $n \geq 3$ the period image $\Phi(\mathcal{W}_n^{\mathrm{st}})$ is a proper subvariety of $O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n$ and need not meet all Hodge loci. We close this gap in Section 4: Kodaira–Spencer transversality produces stable elliptic surfaces at Faltings height n with $\mathrm{rk}(E/K) \geq r$ for every

$$1 \;\leq\; r \;\leq\; \left\lfloor \frac{10n-2}{n-1} \right\rfloor = \left\lfloor \frac{\dim \mathcal{W}_n^{\mathrm{st}}}{h^{2,0}} \right\rfloor \qquad \text{(Theorem 4.4)},$$

the ratio of the moduli dimension $\dim \mathcal{W}_n^{\min} = \dim \mathcal{W}_n^{\mathrm{st}} = 10n-2$ to the cost $h^{2,0} = n-1$ of each Heegner condition.

Remark 3.6. At height $n = 3$, $M_{16}(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C} \cdot E_{16} \oplus \mathbb{C} \cdot \Delta E_4$ is two-dimensional, and the cohomology-valued modular form of Theorem 3.5(1) decomposes as

$$\theta_{\mathrm{KM}}(\tau) \;=\; E_{16}(\tau) \otimes \eta_{\mathrm{Eis}} \;+\; \Delta(\tau) E_4(\tau) \otimes \eta_{\mathrm{cusp}}$$

with $\eta_{\mathrm{Eis}}, \eta_{\mathrm{cusp}} \in H^4(O(\Lambda_3, \mathbb{Z})\backslash\mathcal{D}_3, \mathbb{Q})$. The Eisenstein component η_{Eis} is nonzero since it contains $c_{\mathrm{top}}(\mathcal{F}^\vee)$ as the constant term. The cuspidal component η_{cusp} is nonzero if and only if $L(\Delta E_4, 8) \neq 0$ by the regularized Rallis inner product formula [KR94, GQT14]. The functional equation has sign $+1$, and the LMFDB entry for the newform `1.16.a.a` gives $L(\Delta E_4, 8) \approx 1.521 \neq 0$, so $\eta_{\mathrm{cusp}} \neq 0$. By contrast, at height $n = 2$ one has $\dim S_{10}(\mathrm{SL}_2(\mathbb{Z})) = 0$, forcing $\varphi_{2,\alpha} \propto E_{10}$; the Noether–Lefschetz numbers are then proportional to $\sigma_9(d)$ as computed by Maulik–Pandharipande [MP13].

4. Kodaira–Spencer transversality and geometric realization of Heegner loci

The modularity of the Noether–Lefschetz generating series establishes that the distribution of new algebraic classes by canonical height on $O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n$ is governed by a modular form of weight $6n-2$. However, for $n \geq 3$ (i.e., $p_g \geq 2$) the period map $\Phi\colon \mathcal{W}_n^{\mathrm{st}} \to O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n$ is far from surjective: $\dim \mathcal{W}_n^{\mathrm{st}} = 10n-2$ whereas $\dim \mathcal{D}_n = (n-1)(10n-2) + \frac{1}{2}(n-1)(n-2)$, the two terms corresponding to dimensions of $\mathrm{Hom}(H^{2,0}, H^{1,1}_{\mathrm{prim}})$ and $\bigwedge^2 (H^{2,0})^*$ respectively. Whether $\mathrm{Im}(\Phi)$ meets a given Heegner locus $\mathcal{H}_d$ is not resolved by the theta correspondence alone.

The expected codimension of the Noether–Lefschetz locus for regular elliptic surfaces with section was established by Cox [Cox90] using the Jacobian ring of the Weierstrass equation. We refine this to transversality to $\gamma^\perp$ for every primitive $\gamma \in \Lambda_n$ with $\gamma^2 < 0$, at every general point of the stable stratum $\mathcal{W}_n^{\mathrm{st}}$, using the

orthogonal basis of $H^{1,1}_{\mathrm{prim}}$ and the diagonalization of the Gauss–Manin connection established by Shepherd-Barron [SB23].

Let $[S] \in \mathcal{W}_n^{\mathrm{st}}$ be a general stable elliptic surface with $12n$ nodal fibers, so that the classifying morphism $\varphi\colon \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ is simply ramified at $N = 10n-2$ points $P_1, \dots, P_N$. While Section 1 considered the moduli over a fixed base curve, the period map is invariant under base automorphisms, so we henceforth pass to the moduli stack of stable elliptic surfaces over an unparameterized $\mathbb{P}^1_{\mathbb{C}}$ by quotienting by $\mathrm{Aut}(\mathbb{P}^1) \cong \mathrm{PGL}_2$, yielding $\dim \mathcal{W}_n^{\mathrm{st}} = 10n-2$ (cf. [PS25, Mir81]).

By [SB23, Thms. 4.10 and 4.11], the ramification points give rise to meromorphic 2-forms $\eta_i \in H^0(S, \Omega_S^2(2E_{P_i}))_{\text{2nd kind}}$ whose cohomology classes form an orthogonal basis of $H^{1,1}_{\mathrm{prim}}(S)$. The branch locus of φ induces an étale local coordinate system $(t_1, \dots, t_N)$ on $\mathcal{W}_n^{\mathrm{st}}$, and for each i the derivative $\nabla_{\partial/\partial t_i}$ maps $H^{2,0}(S)$ into the line $\mathbb{C}\cdot\eta_i \subset H^{1,1}_{\mathrm{prim}}(S)$ with kernel the codimension-1 subspace $H^0(S, \Omega_S^2(-E_{P_i}))$.

Lemma 4.1. *For a general stable elliptic surface $[S] \in \mathcal{W}_n^{\mathrm{st}}$ and distinct i, j, let α_{ij} denote the $[\eta_j]$-coefficient of $\nabla_{\partial/\partial t_j}\eta_i$ in $H^{1,1}_{\mathrm{prim}}$. Then $\alpha_{ij} \neq 0$. Furthermore, for any disjoint subsets of branch-point indices $A, B \subset \{1, \dots, N\}$ with $|A| \leq n-1$ and $|B| \geq 8n+1$, the row vectors $(\alpha_{jm})_{m\in B}$ for $j \in A$ are linearly independent.*

Proof. By [SB23, Thm. 4.11(1)], $\nabla_{\partial/\partial t_j}\eta_i \in H^0(\Omega_S^2(2E_{P_i} + 2E_{P_j}))_{\text{2nd kind}}$, and its class in $H^{1,1}_{\mathrm{prim}}$ is a linear combination of $[\eta_i]$ and $[\eta_j]$. The $[\eta_j]$-coefficient α_{ij} vanishes if and only if the double pole along E_{P_j} is absent, i.e., if $\nabla_{\partial/\partial t_j}\eta_i$ lies in $H^0(\Omega_S^2(2E_{P_i}))_{\text{2nd kind}}$. By the classical theory of Schiffer variations (cf. [Gar49, p. 443]), α_{ij} is proportional to $\omega(P_i)\omega(P_j)/(P_i - P_j)^2$, where $\omega = \mathrm{d}x \wedge \mathrm{d}z/(2y)$ generates $H^{2,0}$ locally. For distinct branch points $P_i \neq P_j$ with $\omega(P_i), \omega(P_j) \neq 0$ (which holds generically for $n \geq 3$), this is nonzero, so $\alpha_{ij} \neq 0$.

Since α_{jm} is proportional to $\omega(P_j)\omega(P_m)/(P_j - P_m)^2$, the linear independence follows by a degree count. Suppose $\sum_{j\in A} c_j \alpha_{jm} = 0$ for all $m \in B$. Factoring out $\omega(P_m) \neq 0$, the rational function

$$F(z) \;=\; \sum_{j\in A} \frac{c_j\,\omega(P_j)}{(z-P_j)^2}$$

vanishes at all $z = P_m$ for $m \in B$. Putting $F(z)$ over a common denominator $\prod_{l\in A}(z-P_l)^2$, the numerator is a polynomial of degree at most $2(|A|-1) \leq 2(n-2) = 2n-4$. However, $F(z)$ has roots at the $|B| \geq 8n+1$ points P_m. Since $8n+1 > 2n-4$ for all $n \geq 3$, the polynomial has more roots than its maximum degree, so the numerator polynomial must be identically zero. Evaluating this identically zero numerator at $z = P_j$ gives $c_j\,\omega(P_j)\prod_{l\in A, l\neq j}(P_j - P_l)^2 = 0$. Since the branch points are distinct and $\omega(P_j) \neq 0$, this forces $c_j = 0$ for all $j \in A$. ∎

Theorem 4.2. *Let $k = \mathbb{C}$, $K = \mathbb{C}(t)$, and $n \geq 3$. For every primitive $\gamma \in \Lambda_n$ with $\gamma^2 < 0$, the period map $\Phi\colon \mathcal{W}_n^{\mathrm{st}} \to O(\Lambda_n, \mathbb{Z})\backslash\mathcal{D}_n$ is transverse to the Heegner locus $\mathcal{H}_d$. Equivalently, the Noether–Lefschetz locus*

$$\Phi^{-1}(\mathcal{H}_d) \;\subset\; \mathcal{W}_n^{\mathrm{st}}$$

is either empty or every irreducible component has codimension exactly $n-1$.

Concretely, the composite

$$F_\gamma \circ \Phi \colon \mathcal{W}_n^{\mathrm{st}} \longrightarrow \mathbb{C}^{n-1}, \qquad [S] \mapsto \big((\gamma,\omega_0),\dots,(\gamma,\omega_{n-2})\big),$$

has rank $n-1$ at every general point of $\Phi^{-1}(\mathcal{H}_d)$, where $\omega_0,\dots,\omega_{n-2}$ is a basis of $H^{2,0}(S)$.

Proof. The Kodaira–Spencer map $\kappa\colon T_{[S]}\mathcal{W}_n^{\mathrm{st}} \to \mathrm{Hom}(H^{2,0}(S), H^{1,1}_{\mathrm{prim}}(S))$ is the $(1,1)$-projection of the Gauss–Manin derivative $\kappa(\partial/\partial t_i)(\omega) = (\nabla_{\partial/\partial t_i}\omega)^{1,1}$ for $\omega \in H^{2,0}(S)$. We first show $\mathrm{Im}(\kappa) \not\subset \gamma^\perp$. At a general surface, γ is not of type $(1,1)$; write its Hodge decomposition $\gamma = \gamma^{2,0}+\gamma^{1,1}+\gamma^{0,2}$ in $\Lambda_n\otimes\mathbb{C}$. The intersection form is positive definite on $H^{2,0}\oplus H^{0,2}$ and negative definite on $H^{1,1}_{\mathrm{prim}}$ by the Hodge–Riemann bilinear relations. Since $\gamma^2<0$, the negative-definite part must dominate, forcing $\gamma^{1,1}\neq 0$. Since $\eta_i \in H^{1,1}_{\mathrm{prim}}$ pairs trivially with $H^{2,0}\oplus H^{0,2}$, we have $(\gamma,\eta_i) = (\gamma^{1,1},\eta_i)$, and the orthogonality of the η_i gives $(\gamma,\eta_i)\neq 0$ for some i. Choosing $\omega \in H^{2,0}(S) \smallsetminus H^0(\Omega_S^2(-E_{P_i}))$ so that $\kappa(\partial/\partial t_i)(\omega) = c_i(\omega)\,\eta_i$ with $c_i(\omega)\neq 0$,

$$\big(\gamma,\ \kappa(\partial/\partial t_i)(\omega)\big) \;=\; c_i(\omega)\cdot(\gamma,\eta_i) \;\neq\; 0.$$

For the full rank statement, we compute the Jacobian of $F_\gamma\circ\Phi$ on the Heegner locus. Since γ is a flat section of the local system, differentiating the period integral $f_j = (\gamma,\omega_j)$ gives $\partial_{t_i} f_j = (\gamma, \nabla_{\partial/\partial t_i}\omega_j)$. By Griffiths transversality, the Gauss–Manin derivative $\nabla_{\partial/\partial t_i}\omega_j$ decomposes into an $H^{2,0}$ part and an $H^{1,1}$ part; by [SB23, Thm. 4.12(1)] the $H^{1,1}$ part is proportional to η_i. The $H^{2,0}$ part pairs with γ to give a linear combination of the period integrals (γ,ω_k), all of which vanish on the Heegner locus $\gamma\perp H^{2,0}$. The Jacobian therefore reduces to

$$\partial_{t_i} f_j\big|_{\gamma\perp H^{2,0}} \;=\; c_{ij}\cdot(\gamma,\eta_i),$$

where $c_{ij}\neq 0$ precisely when ω_j does not vanish at P_i. Taking the monomial basis $\omega_j = z^j\cdot \mathrm{d}x\wedge\mathrm{d}z/(2y)$ of $H^{2,0}(S)$ and setting $\omega_0 = \mathrm{d}x\wedge\mathrm{d}z/(2y)$, the Gauss–Manin derivative acts as $\nabla_{\partial/\partial t_i}(\omega_j) = z^j\nabla_{\partial/\partial t_i}(\omega_0)$, which evaluates at the singular fiber E_{P_i} to give the exact factorization $c_{ij} = c_i(\omega_0)P_i^j$. On the Heegner locus γ is of type $(1,1)$, hence $\gamma=\gamma^{1,1}\neq 0$ in $H^{1,1}_{\mathrm{prim}}$; since the η_i span $H^{1,1}_{\mathrm{prim}}$, at every point of $\Phi^{-1}(\mathcal{H}_d)$ at least one pairing $g_i(S) := (\gamma,\eta_i(S))$ is nonzero.

We claim that at a general point at least $n-1$ pairings are nonzero. Suppose for contradiction that fewer than $n-1$ pairings are nonzero on some irreducible component V_0 of $\Phi^{-1}(\mathcal{H}_d)$. Let

$$Z \;:=\; \{m\in\{1,\dots,N\} : g_m \equiv 0 \text{ on } V_0\},$$

so that $|Z|\geq N-(n-2) = 9n$ by assumption, and γ expands on V_0 as $\gamma = \sum_{j\notin Z} c_j\eta_j$ with at most $n-2$ nonzero terms.

The vanishing $g_m\equiv 0$ on V_0 implies that $dg_m|_p$ lies in the conormal space $N_p^*V_0$ at every general smooth point $p\in V_0$, and by Krull's height theorem (or equivalently [Voi03, Prop. 5.14]) $\dim N_p^*V_0 \leq n-1$. We compute the structure of dg_m for $m\in Z$. By the Leibniz rule and orthogonality of the η-basis, $\partial_{t_i} g_m = (\gamma,\nabla_{\partial/\partial t_i}\eta_m)$ is supported on the indices $i=m$ and $i\notin Z$: indeed, for $i\in Z\smallsetminus\{m\}$, [SB23,

Thm. 4.11(1)] expresses $\nabla_{\partial/\partial t_i}\eta_m$ as a linear combination of η_i and η_m, both of which pair to zero against γ on V_0. Writing $dg_m = a_m\,dt_m + w_m$ with $w_m \in W := \operatorname{span}\{dt_i : i \notin Z\}$, the coefficient is

$$a_m \;=\; -(\eta_m,\eta_m)\sum_{j\notin Z} c_j\alpha_{jm}.$$

In the quotient T^*_p/W, the projections $a_m\,dt_m$ for distinct $m \in Z$ are linearly independent, yet all lie in the image of $N^*_pV_0$, which has dimension $\le n-1$. Hence at most $n-1$ of the a_m are nonzero, so the subset

$$Z' \;:=\; \{m \in Z : a_m = 0\} \;=\; \Big\{m \in Z : \textstyle\sum_{j\notin Z} c_j\alpha_{jm} = 0\Big\}$$

has $|Z'| \ge |Z| - (n-1) \ge 8n+1$. Applying Lemma 4.1 with $A = \{1,\dots,N\} \smallsetminus Z$ (of size $\le n-2 < n-1$) and $B = Z'$ (of size $\ge 8n+1$) forces $c_j = 0$ for all $j \notin Z$, giving $\gamma = 0$ and contradicting $\gamma^2 < 0$. Therefore at least $n-1$ pairings g_i are not identically zero on V_0. For these pairings, the zero locus $\{g_i = 0\} \cap V_0$ is proper, so at a general point of $\Phi^{-1}(\mathcal{H}_d)$ at least $n-1$ pairings are simultaneously nonzero.

Choosing $n-1$ indices $i_1,\dots,i_{n-1}$ with $(\gamma,\eta_{i_k}) \neq 0$, the $(n-1)\times(n-1)$ Jacobian matrix $J_{j,k} = \partial_{t_{i_k}} f_j$ factors exactly as $\operatorname{Vand}(P_{i_1},\dots,P_{i_{n-1}})\cdot\operatorname{diag}\big(c_{i_k}(\omega_0)(\gamma,\eta_{i_k})\big)$. Since the base form ω_0 generically does not vanish at the branch points, $c_{i_k}(\omega_0) \neq 0$. Both factors are therefore nonsingular, so $d(F_\gamma \circ \Phi)$ has rank $n-1$ at every general point of $\Phi^{-1}(\mathcal{H}_d)$, and the implicit function theorem gives the expected codimension.

■

Corollary 4.3. *For every $n \ge 3$, there exist infinitely many integers $d \ge 1$ for which there is a stable elliptic surface $\pi\colon S \to \mathbb{P}^1$ at Faltings height n such that* $\operatorname{rk}(E_S/K) \ge 1$ *and the Mordell–Weil lattice contains a section of canonical height $\hat{h}(P) = d$.*

Proof. Theorem 4.2 establishes that the period map is nonconstant. By Cattani–Deligne–Kaplan [CDK95], the Noether–Lefschetz locus $\bigcup_{d\ge1}\Phi^{-1}(\mathcal{H}_d)$ is a countable union of closed algebraic subvarieties. To show $\Phi^{-1}(\mathcal{H}_d) \neq \varnothing$ for infinitely many d, fix $[S] \in \mathcal{W}_n^{\mathrm{st}}$ very general, with geometric Picard rank 2 so that $\Lambda_n \cap H^{1,1}_{\mathrm{prim}}(S) = \{0\}$. Let $V_{\mathbb{R}} = (H^{2,0}\oplus H^{0,2}) \cap H^2(S,\mathbb{R})$, of dimension $2(n-1)$.

We claim that the orthogonal projection of Λ_n to $V_{\mathbb{R}}$ has dense image. Suppose otherwise: the closure of the projection is a proper closed subgroup of $V_{\mathbb{R}}$. By the structure theorem for closed subgroups of $\mathbb{R}^d$, this closure has the form $W \oplus L$ where $W \subset V_{\mathbb{R}}$ is a proper linear subspace and $L \subset W^\perp$ is a discrete subgroup (a lattice in some subspace of $W^\perp$). Hence there exists a nonzero linear functional ℓ on $V_{\mathbb{R}}$ that vanishes on W and takes integer values on L, hence integer values on the closure, hence on the projection of Λ_n. Since Q is positive-definite on $V_{\mathbb{R}}$, this functional is $\ell = Q(\cdot,\alpha)$ for some nonzero $\alpha \in V_{\mathbb{R}}$. Extending α by zero on $H^{1,1}_{\mathrm{prim},\mathbb{R}}$, the condition $Q(\gamma,\alpha) \in \mathbb{Z}$ for all $\gamma \in \Lambda_n$ places α in the dual lattice Λ_n^*, hence $\alpha \in \Lambda_n \otimes \mathbb{Q}$.

Thus α is a nonzero rational class in $V_{\mathbb{R}} \subset H^{2,0}(S)\oplus H^{0,2}(S)$, i.e., its $(1,1)$-component in the Hodge decomposition of S vanishes. For each fixed nonzero

$\alpha \in \Lambda_n \otimes \mathbb{Q}$, the locus

$$\{[S'] \in \mathcal{D}_n : \alpha^{1,1}(S') = 0\}$$

is a proper analytic subvariety of the period domain, defined by the closed condition $\alpha \in F^2(S') + \overline{F^2(S')}$ on the holomorphically varying Hodge filtration. It is proper because the monodromy representation on $\Lambda_n \otimes \mathbb{Q}$ is irreducible (on $\mathcal{W}_n^{\mathrm{st}}$ all singular fibers are nodal, so the Picard–Lefschetz irreducibility theorem [Voi03, Ch. 3, Thm. 3.27] applies): no nonzero rational class can stay in the proper subspace $V_{\mathbb{R}}$ throughout the period image. Since $[S]$ is very general — outside the countable union, over all nonzero $\alpha \in \Lambda_n \otimes \mathbb{Q}$, of these proper subvarieties — no such α exists, contradicting the construction of α in the previous paragraph. The projection of Λ_n to $V_{\mathbb{R}}$ is therefore dense, and Λ_n contains sequences of primitive vectors γ with $|f_j(\gamma)| = |(\gamma, \omega_j)| \to 0$ for all j, as $d = -\gamma^2/2 \to \infty$.

We construct sequences with prescribed limit direction. Setting $\tilde{f}_j = f_j/\|\gamma\|$ does not change the zero set. By density of the $V_{\mathbb{R}}$-projection (previous paragraph), there exist primitive $\gamma \in \Lambda_n$ with $V_{\mathbb{R}}$-projection arbitrarily small; and the set of normalized primitive vectors $\gamma/\|\gamma\|$ is dense in the unit sphere of $H^2_{\mathrm{prim}}(S, \mathbb{R})$ since Λ_n has full rank. Combining the two, we choose primitive $\gamma_k \in \Lambda_n$ with $|(\gamma_k, \omega_j)| \to 0$ for all j, $\|\gamma_k\| \to \infty$, and $\gamma_k/\|\gamma_k\| \to \hat{\gamma}_\infty$ for any prescribed unit direction $\hat{\gamma}_\infty \in H^{1,1}_{\mathrm{prim}}(S)_{\mathbb{R}}$.

The asymptotic Jacobian of $\tilde{f}$ at $[S]$, taken with respect to a choice of $n-1$ branch-point coordinates $t_{i_1}, \dots, t_{i_{n-1}}$, is by Theorem 4.2 the $(n-1)\times(n-1)$ matrix

$$J^\infty_{j,k} \;=\; c_{i_k}(\omega_0) \cdot P^j_{i_k} \cdot (\hat{\gamma}_\infty, \eta_{i_k}),$$

which factors as a Vandermonde times a diagonal. Choosing $\hat{\gamma}_\infty$ outside the finitely many hyperplanes $\eta_{i_k}^\perp$ in $H^{1,1}_{\mathrm{prim}}(S)_{\mathbb{R}}$ makes each $(\hat{\gamma}_\infty, \eta_{i_k})$ nonzero, so J^∞ is nonsingular.

We apply Smale's α-theorem [Sma86, Thm. A] (a quantitative implicit function theorem) to $\tilde{f}$ at $[S]$ for $k \gg 0$. The theorem gives an actual zero $[S_k] \in \mathcal{W}_n^{\mathrm{st}}$ of $\tilde{f}$ near $[S]$ provided the Smale invariant α — the product of the Newton step length β and the higher-derivative invariant of $\tilde{f}$ at $[S]$ — is below the universal threshold α_0. The Newton step is bounded by

$$\beta \;=\; \|D\tilde{f}([S])^{-1}\tilde{f}([S])\| \;\le\; \|D\tilde{f}([S])^{-1}\| \cdot \|\tilde{f}([S])\|,$$

where $\|D\tilde{f}([S])^{-1}\|$ is uniformly bounded in k since $D\tilde{f}([S]) \to J^\infty$ is nonsingular, and $\|\tilde{f}([S])\| \to 0$ since $|f_j(\gamma_k)| \to 0$ and $\|\gamma_k\| \to \infty$. The higher-derivative invariant is uniformly bounded by analyticity of the period map on a compact neighborhood of $[S]$. Hence $\alpha \to 0$, eventually below α_0, and Smale's theorem produces $[S_k]$ near $[S]$ with $\tilde{f}([S_k]) = 0$. This gives $\Phi^{-1}(\mathcal{H}_{d_k}) \neq \varnothing$ for $d_k = -\gamma_k^2/2$.

By Theorem 4.2, each nonempty $\Phi^{-1}(\mathcal{H}_d)$ has codimension exactly $n-1$. On the stable stratum $R(\mathfrak{f}) = 0$, so every newly algebraic class $\gamma_k \in \Lambda_n$ with $\gamma_k^2 = -2d_k$ projects nontrivially to $\mathrm{MW}(E_{S_k}/K)$ by Theorem 3.5(2), giving $\mathrm{rk}(E_{S_k}/K) \ge 1$ and a section of canonical height $\hat{h}(P) = d_k$. ∎

4.1. **Higher Mordell–Weil rank.** The argument of Corollary 4.3 extends to r simultaneous algebraic classes.

Theorem 4.4. *Let $n \geq 3$. For every integer*

$$1 \;\leq\; r \;\leq \left\lfloor \frac{10n-2}{n-1} \right\rfloor = \left\lfloor \frac{\dim \mathcal{W}_n^{\mathrm{st}}}{h^{2,0}(S)} \right\rfloor,$$

there exist infinitely many stable elliptic surfaces $\pi\colon S \to \mathbb{P}^1$ at Faltings height n with Mordell–Weil rank $\mathrm{rk}(E_S/K) \geq r$.

Proof. We seek a surface $[S_m] \in \mathcal{W}_n^{\mathrm{st}}$ near a general $[S]$ at which r independent primitive classes $\gamma_1, \ldots, \gamma_r \in \Lambda_n$ become simultaneously Hodge classes; equivalently, the $M = r(n-1)$ period equations

$$f_{a,j} = (\gamma_a, \omega_j) = 0, \qquad 1 \leq a \leq r,\ 0 \leq j \leq n-2,$$

are simultaneously satisfied. By the hypothesis $r \leq \lfloor (10n-2)/(n-1) \rfloor$, we have $M \leq 10n-2 = \dim \mathcal{W}_n^{\mathrm{st}}$. Choose M branch-point coordinates $t_{i_1}, \ldots, t_{i_M}$ to vary (fixing the remaining coordinates at $[S]$); the period equations $\{f_{a,j}\}$ then form an $M \times M$ system whose Jacobian is square.

The construction of Corollary 4.3 produces, for each prescribed limit direction in $H^{1,1}_{\mathrm{prim}}(S)_{\mathbb{R}}$, a sequence of primitive vectors $\gamma_m \in \Lambda_n$ with $|f_j(\gamma_m)| \to 0$, $\|\gamma_m\| \to \infty$, and $\gamma_m/\|\gamma_m\|$ converging to that direction. The genericity condition on $[S]$ does not depend on the choice of limit direction, so the same $[S]$ supports the construction for every limit direction in $H^{1,1}_{\mathrm{prim}}(S)_{\mathbb{R}}$. Each construction produces infinitely many primitive vectors, so for r distinct limit directions $\hat{\gamma}_{1,\infty}, \ldots, \hat{\gamma}_{r,\infty} \in H^{1,1}_{\mathrm{prim}}(S)_{\mathbb{R}}$ we may choose the sequences $\gamma_{a,m}$ pairwise disjoint while preserving the limit properties.

We now construct the limit Jacobian. Normalize the period equations by $\tilde{f}_{a,j} = f_{a,j}/\|\gamma_{a,m}\|$ (same zero set). The Jacobian $D\tilde{f}([S])$ is an $M \times M$ matrix whose rows are indexed by the equations $\tilde{f}_{a,j}$ (with $1 \leq a \leq r$ and $0 \leq j \leq n-2$) and whose columns are indexed by the variables t_{i_k} (with $1 \leq k \leq M$). The entry in row (a,j) and column k converges, as $m \to \infty$, to

$$J^{\infty}_{(a,j),k} \;=\; c_{i_k}(\omega_0) \cdot P_{i_k}^{\,j} \cdot (\hat{\gamma}_{a,\infty}, \eta_{i_k}),$$

by Theorem 4.2. We must show that $\det J^{\infty} \neq 0$ for some choice of limit directions.

The determinant $\Delta := \det J^{\infty}$ is a polynomial in the pairing variables $X_{a,k} = (\hat{\gamma}_a, \eta_{i_k})$, where $\hat{\gamma}_a$ now ranges over $H^{1,1}_{\mathrm{prim}}(S, \mathbb{C})$. Partition $\{1, \ldots, M\}$ into r disjoint blocks $C_1, \ldots, C_r$ of size $n-1$, and set $\hat{\gamma}_a = \sum_{k \in C_a} \eta_{i_k}$ for each a. By orthogonality of the η-basis, this gives $X_{a,k} = (\eta_{i_k}, \eta_{i_k})$ for $k \in C_a$ and 0 otherwise, so J^{∞} is block-diagonal with r blocks of size $(n-1) \times (n-1)$. The a-th block has determinant

$$\mathrm{Vand}\big(\{P_{i_k}\}_{k \in C_a}\big) \cdot \prod_{k \in C_a} c_{i_k}(\omega_0)\,(\eta_{i_k}, \eta_{i_k}),$$

which is nonzero generically: the Vandermonde by distinctness of branch points, the $c_{i_k}(\omega_0)$ by non-vanishing of $\omega_0 \in H^0(\mathcal{O}(n-2))$ at the branch points (a generic condition for $n \geq 3$), and the (η_{i_k}, η_{i_k}) by Hodge–Riemann negative-definiteness. Hence Δ is not the zero polynomial. Since the pairing variables $X_{a,k}$ restrict to real-valued coordinates on the real form $(H^{1,1}_{\mathrm{prim}}(S)_{\mathbb{R}})^r$, the polynomial Δ does not vanish identically there, so its non-vanishing locus on the real form is Zariski-open dense. Choose $\mathbb{R}$-linearly independent $\hat{\gamma}_{1,\infty}, \ldots, \hat{\gamma}_{r,\infty}$ in this open subset.

For this choice of real sequences, Smale's α-theorem [Sma86, Thm. A] applies to $\tilde{f}$ at $[S]$ by the same bounds as in Corollary 4.3: the Newton step $\beta \to 0$ since $\|D\tilde{f}([S])^{-1}\|$ is uniformly bounded in m (by nonsingularity of J^∞) and $\|\tilde{f}([S])\| \to 0$, while the higher-derivative invariant is uniformly bounded by analyticity of the period map. Hence $\alpha < \alpha_0$ for $m \gg 0$, and Smale's theorem produces a zero $[S_m] \in \mathcal{W}_n^{\mathrm{st}}$ of $\tilde{f}$ near $[S]$.

At $[S_m]$, each $\gamma_{a,m}$ is a Hodge class. The vectors $\gamma_{a,m}/\|\gamma_{a,m}\|$ converge to the $\mathbb{R}$-linearly independent directions $\hat{\gamma}_{a,\infty}$, so for $m \gg 0$ the $\gamma_{a,m}$ are $\mathbb{R}$-linearly independent in $\Lambda_n \otimes \mathbb{R}$, hence $\mathbb{Q}$-linearly independent. On the stable stratum $R(\mathfrak{f}) = 0$, so by Theorem 3.5(2) each $\gamma_{a,m}$ projects nontrivially to $\mathrm{MW}(E_{S_m}/K)$. The r classes therefore generate a rank-r sublattice of $\mathrm{MW}(E_{S_m}/K)$, giving $\mathrm{rk}(E_{S_m}/K) \geq r$. ∎

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