

ANALYTICALLY DENSE MORDELL–WEIL RANK JUMPS ON ELLIPTIC SURFACES VIA TRANSVERSE REALIZATION

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In memory of Michael Atiyah, 1929–2019

ABSTRACT. Following Park–Schmitt, let $\mathcal{W}_{\min,n}$ denote the moduli stack of minimal Weierstrass fibrations of Faltings height $n \geq 2$ over an unparameterized $\mathbb{P}_{\mathbb{C}}^1$. A very general member has Mordell–Weil rank zero. For every integer r satisfying

$$1 \leq r \leq \left\lfloor \frac{10n-2}{n-1} \right\rfloor$$

we prove that the locus of simply-branched elliptic surfaces with only I_1 singular fibres and Mordell–Weil rank at least r is analytically dense in $\mathcal{W}_{\min,n}$. The locus of such surfaces with rank exactly one is also analytically dense.

More precisely, within the simply-branched all- I_1 locus, every point is an analytic limit of pairwise non-isomorphic Jacobian elliptic surfaces carrying r independent sections whose canonical heights all tend to infinity. These sections arise from individually primitive integral $(1, 1)$ classes imposing $r(n-1)$ independent period conditions. After base change to a marked deformation chart, the deformation germ of each constructed surface with its ordered sections is identified with the corresponding smooth Hodge-locus germ. The proof combines Shepherd–Barron’s infinitesimal period calculation with a block Vandermonde construction, lattice approximation, and the holomorphic implicit function theorem.

1. INTRODUCTION

The global sections of a Jacobian elliptic surface form its Mordell–Weil group. At each fixed Faltings height $n \geq 2$, a very general such surface over $\mathbb{P}_{\mathbb{C}}^1$ has rank zero. We study how additional sections appear under arbitrarily small deformations at the same height. We establish analytic density of Jacobian elliptic surfaces with Mordell–Weil rank at least r for every positive integer r in an explicit range, and of such surfaces with rank exactly one. Our construction illustrates the emergence of global sections by coordinating local branch deformations through global period equations, establishing transverse intersections of the elliptic surface period image with suitable simultaneous Hodge loci.

Throughout the paper we work over $\mathbb{C}$. By a Jacobian elliptic surface over $\mathbb{P}^1$ we mean a smooth projective surface S equipped with a morphism $\pi: S \rightarrow \mathbb{P}^1$ with connected fibres and smooth generic fibre of genus one, together with a distinguished section $\sigma: \mathbb{P}^1 \rightarrow S$, called the zero section. We assume that π is relatively-minimal, meaning that no fibre contains a (-1) -curve. Write $\omega_{S/\mathbb{P}^1}$ for the relative dualizing sheaf. Its fundamental line bundle and Faltings height are

$$\mathcal{L} = (R^1\pi_*\mathcal{O}_S)^\vee \simeq \pi_*\omega_{S/\mathbb{P}^1}, \quad n = \deg \mathcal{L} = \chi(\mathcal{O}_S).$$

2020 *Mathematics Subject Classification*. Primary 14J27; Secondary 14C30, 14D07.

Key words. Elliptic surfaces, Mordell–Weil rank, period maps, Noether–Lefschetz loci.

We assume $n \geq 2$. The minimal discriminant divisor has degree $12n$; see [Mir89, BHPVdV04, SS10] for background on elliptic surfaces and their invariants.

Let $\mathcal{W}_{\min,n}$ denote the moduli stack of minimal Weierstrass fibrations of height n over an unparameterized $\mathbb{P}^1$. It is obtained from the fixed-base height moduli stack of minimal Weierstrass fibrations with smooth generic fibre by quotienting by changes of coordinates on the base [PS25, Thms. 1.1–1.2]. Since the base $\mathbb{P}^1$ is unparameterized, isomorphisms may change its coordinates but must carry fibres to fibres and preserve the zero section. The fixed-base height moduli description by minimal weighted linear series is given in [BPS24, Prop. 7.5]. We identify the geometric points of $\mathcal{W}_{\min,n}$ with relatively-minimal Jacobian elliptic surfaces by taking the minimal resolution of each Weierstrass model. Analytic density is understood after pullback to analytic atlases; equivalently, it can be checked on the coarse moduli spaces with their analytic topology.

Write $E_S/\mathbb{C}(t)$ for the generic fibre and

$$\mathrm{MW}(S/\mathbb{P}^1) = E_S(\mathbb{C}(t))$$

for the Mordell–Weil group of sections. Let F and O denote the fibre and zero-section classes, and let $R(S)$ be the root lattice generated by the components of reducible fibres that do not meet O . Let $\mathrm{NS}(S)$ denote the Néron–Severi group of S . The trivial lattice $\mathrm{Triv}(S) \subset \mathrm{NS}(S)$ is generated by F , O , and these components. The Shioda–Tate description [Shi90, Thm. 1.3] gives

$$0 \longrightarrow \mathrm{Triv}(S) \longrightarrow \mathrm{NS}(S) \longrightarrow \mathrm{MW}(S/\mathbb{P}^1) \longrightarrow 0.$$

Writing $\rho(S) = \mathrm{rk} \mathrm{NS}(S)$, we obtain

$$(1) \quad \rho(S) = 2 + \mathrm{rk} R(S) + \mathrm{rk} \mathrm{MW}(S/\mathbb{P}^1).$$

The root-lattice term is determined by the Kodaira types of the singular fibres. To remove this contribution, we work on the squarefree-discriminant locus $\mathcal{W}_n^{\mathrm{I}_1} \subset \mathcal{W}_{\min,n}$. Every singular fibre on this locus is of Kodaira type I_1 , that is, an irreducible rational curve with a single ordinary node. Thus $R(S) = 0$, and (1) becomes

$$\mathrm{rk} \mathrm{MW}(S/\mathbb{P}^1) = \rho(S) - 2.$$

On this locus, algebraic classes that are linearly independent modulo $\langle F, O \rangle$ give linearly independent Mordell–Weil sections.

To describe these classes, put

$$\Lambda_n = \langle F, O \rangle^\perp \subset H^2(S, \mathbb{Z})/\mathrm{tors}.$$

Its isomorphism type depends only on n ; we use the same symbol for a fixed abstract model. Throughout the paper, integral cohomology lattices are understood modulo torsion. For a lattice or finitely generated abelian group A , write $A_K = A \otimes_{\mathbb{Z}} K$ for $K \in \{\mathbb{Q}, \mathbb{R}, \mathbb{C}\}$. Our primitive Hodge-space conventions are

$$\begin{aligned} H_{\mathrm{prim}}^{1,1}(S) &= H^{1,1}(S) \cap \Lambda_{n,\mathbb{C}}, \\ H_{\mathrm{prim}}^{1,1}(S, \mathbb{R}) &= H^{1,1}(S) \cap \Lambda_{n,\mathbb{R}}. \end{aligned}$$

We use the half-normalized Néron–Tate pairing

$$\langle P, Q \rangle_{\hat{h}} = \frac{1}{2} \langle P, Q \rangle_{\mathrm{Sh}}, \quad \hat{h}(P) = \langle P, P \rangle_{\hat{h}},$$

where $\langle, \rangle_{\text{Sh}}$ is Shioda’s lattice height pairing. On the all- I_1 locus, a non-zero section P satisfies $\hat{h}(P) = n + P \cdot O$, where $P \cdot O$ counts its intersections with the zero section, with multiplicity; see Proposition 3.16.

The moduli stack $\mathcal{W}_n^{I_1}$ is irreducible of dimension $10n - 2$, as follows from the quotient description in [PS25, Thms. 1.1 and 1.2(a)]. Each Jacobian elliptic surface in this locus has a classifying morphism

$$\varphi_S: \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}.$$

Here $\overline{\mathcal{M}}_{1,1}$ is the moduli stack of stable genus one curves with one marked point, and $\mathcal{M}_{1,1}$ is its open substack parametrizing smooth curves. This morphism is unramified over the boundary $\overline{\mathcal{M}}_{1,1} \setminus \mathcal{M}_{1,1}$. Its stack-theoretic ramification divisor $\text{Ram}(\varphi_S)$ is the zero divisor of its differential; it differs from the ramification divisor of the coarse j -map. We call S *simply-branched* when $\text{Ram}(\varphi_S)$ is reduced, and denote the resulting substack by

$$\mathcal{W}_n^{\text{sb}} \subset \mathcal{W}_n^{I_1}.$$

Distinct ramification points are allowed to have the same image under φ_S . Lemmas 3.2 and 3.3 show that $\mathcal{W}_n^{\text{sb}}$ is Zariski open and dense in $\mathcal{W}_{\min,n}$, hence also analytically dense.

Elliptic $K3$ surfaces, which constitute the Calabi–Yau case $n = 2$, provide the archetype for our question. Since $F \cdot F = 0$, $F \cdot O = 1$, and $O^2 = -2$, the classes $F, O + F$ identify $\langle F, O \rangle$ with the hyperbolic plane U . We recall a standard consequence of local Torelli and the Shioda–Tate formula, including the proof to explain the comparison with higher height $n \geq 3$.

Theorem 1.1. *The locus*

$$\{[S] \in \mathcal{W}_2^{I_1} : \text{rk MW}(S/\mathbb{P}^1) = 18\}$$

is analytically dense in $\mathcal{W}_2^{I_1}$. Consequently, for every $1 \leq r \leq 18$, the locus of Jacobian elliptic surfaces with Mordell–Weil rank at least r is analytically dense in $\mathcal{W}_2^{I_1}$.

Proof. Fix $[S] \in \mathcal{W}_2^{I_1}$. The lattice $\Lambda_2 = \langle F, O \rangle^\perp$ has signature $(2, 18)$. By local Torelli, the marked deformations of S on which F and O remain algebraic are locally parametrized by an open subset of a connected component $\mathcal{D}_2^+$ of

$$\{\Pi \subset \Lambda_{2,\mathbb{R}} : \Pi \text{ is an oriented positive two-plane}\};$$

see [Huy16, Ch. 6, Prop. 2.8 and §2.4]. The line bundles $\mathcal{O}_S(F)$ and $\mathcal{O}_S(O)$ extend over this fixed-class deformation space. On the central surface, their spaces of sections have dimensions two and one, respectively, and their first cohomology groups vanish. Cohomology and base change therefore extend the pencil and its zero-section divisor. After shrinking, the pencil remains basepoint free and the divisor remains a smooth rational curve; their intersection number one makes this divisor a section. The squarefree discriminant condition is open, so we obtain a marked chart B in $\mathcal{W}_2^{I_1}$ whose period image is open in $\mathcal{D}_2^+$.

Positive two planes defined over $\mathbb{Q}$ are dense in $\mathcal{D}_2^+$. Indeed, a positive ordered basis of any such real plane can be approximated by vectors in $\Lambda_{2,\mathbb{Q}}$, and sufficiently close approximations still span positive planes with the induced orientation.

Choose a rational plane Π in the period image of B , and let S_Π be the corresponding surface.

Since Π is defined over $\mathbb{Q}$, its orthogonal complement is a rational subspace of dimension 18. Hence

$$\mathrm{rk}(\Pi^\perp \cap \Lambda_2) = 18.$$

These integral classes are of type $(1, 1)$ on S_Π , so the Lefschetz $(1, 1)$ -theorem makes them algebraic. Together with F and O , these give twenty independent classes in $\mathrm{NS}(S_\Pi)$. Since divisor classes lie in $H^{1,1}(S_\Pi)$, we have

$$20 \leq \rho(S_\Pi) \leq h^{1,1}(S_\Pi) = 20.$$

Hence $\rho(S_\Pi) = 20$.

The surface S_Π lies in $\mathcal{W}_2^{l_1}$, so its singular fibres are irreducible. Thus (1) gives

$$\mathrm{rk} \mathrm{MW}(S_\Pi/\mathbb{P}^1) = \rho(S_\Pi) - 2 = 18.$$

Rational positive planes occur in every open subset of the local period image, proving density. The assertion for rank at least r follows immediately. $\blacksquare$

This is the classical period-theoretic density argument with the fibre and zero-section classes fixed. For related results, Oguiso proves density of every prescribed Picard number in the full Kuranishi space and constructs a hierarchy of narrow Mordell–Weil lattices of Jacobian $K3$ surfaces [Ogu03, Cor. 1.3 and Thm. 1.4].

In the $K3$ case, local Torelli identifies the marked deformation space preserving F and O with an open subset of $\mathcal{D}_2^+$. At higher height, the dimension of the ambient period domain prevents such local openness. The primitive Hodge numbers are

$$h^{2,0} = h^{0,2} = n - 1, \quad h_{\mathrm{prim}}^{1,1} = 10n - 2.$$

For general n , let $\mathcal{D}_n^+$ denote the connected component of the corresponding polarized weight-2 period domain containing the marked periods under consideration. For $n \geq 3$, its dimension satisfies

$$\dim_{\mathbb{C}} \mathcal{D}_n^+ = (n - 1)(10n - 2) + \binom{n - 1}{2} > 10n - 2 = \dim \mathcal{W}_n^{l_1}.$$

The local elliptic surface period image therefore has positive codimension in $\mathcal{D}_n^+$. Density of Hodge loci in the ambient domain alone no longer guarantees that they meet this image.

A fixed class $\gamma \in \Lambda_n$ with $\gamma^2 < 0$ defines a Hodge locus of complex codimension $n - 1$ in $\mathcal{D}_n^+$. After restriction to the elliptic surface deformation space, however, its defining equations need not remain independent. We must establish the required intersections using deformations of Jacobian elliptic surfaces themselves.

On a small simply-connected deformation chart, a marking identifies the primitive cohomology lattices of nearby surfaces with the fixed lattice Λ_n by parallel transport. For fixed real classes $\gamma_1, \dots, \gamma_r \in \Lambda_{n, \mathbb{R}}$, their simultaneous Hodge locus is defined by

$$\gamma_a \perp F^2 \quad (1 \leq a \leq r),$$

where $F^2 = H^{2,0}$ varies with the surface. Since the classes are real, these equations also imply orthogonality to the conjugate subspace. They therefore say exactly that every γ_a is of type $(1, 1)$.

Our main theorem constructs independent Mordell–Weil sections whose canonical heights tend to infinity while the Faltings height $n \geq 2$ of the surfaces remains fixed.

Main Theorem 1.2. *Let $n \geq 2$ and $r \geq 1$ be integers satisfying*

$$r(n-1) \leq 10n-2.$$

Let $\mathcal{W}_{\min,n}$ denote the complex moduli stack of minimal Weierstrass fibrations of Faltings height n over an unparameterized $\mathbb{P}_{\mathbb{C}}^1$. Let $\mathcal{W}_n^{\text{sb}} \subset \mathcal{W}_{\min,n}$ be the locus where every singular fibre is of Kodaira type I_1 and the differential of the classifying morphism

$$\varphi_S: \mathbb{P}_{\mathbb{C}}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}$$

has reduced zero divisor. For every $[S] \in \mathcal{W}_n^{\text{sb}}$, there are pairwise non-isomorphic Jacobian elliptic surfaces

$$[S_m] \in \mathcal{W}_n^{\text{sb}}, \quad [S_m] \longrightarrow [S]$$

in the analytic topology, carrying independent sections $P_{1,m}, \dots, P_{r,m}$ in their Mordell–Weil groups.

More precisely, these surfaces occur in a simply-connected analytic étale local deformation chart $B \rightarrow \mathcal{W}_n^{\text{sb}}$ of dimension $10n-2$, with $S_{b_0} = S$, $S_{b_m} = S_m$, and $b_m \rightarrow b_0$. A marking identifies the lattices $\langle F_b, O_b \rangle^{\perp} \subset H^2(S_b, \mathbb{Z})/\text{tors}$ with a fixed lattice Λ_n by parallel transport preserving the cup-product pairing, where F_b, O_b are the fibre and zero-section classes. The sections $P_{a,m}$ are the images of individually primitive, linearly independent classes $\gamma_{1,m}, \dots, \gamma_{r,m} \in \Lambda_n$, of type $(1,1)$ on S_m , under $\text{NS}(S_m)/\langle F_{b_m}, O_{b_m} \rangle \simeq \text{MW}(S_m/\mathbb{P}^1)$. With canonical heights normalized as half Shioda’s lattice heights,

$$\widehat{h}(P_{a,m}) = -\frac{1}{2}\gamma_{a,m}^2 \longrightarrow \infty \quad (1 \leq a \leq r).$$

Choose a local holomorphic frame $\omega_0(b), \dots, \omega_{n-2}(b)$ for $H^{2,0}(S_b)$, viewed in $\Lambda_n \otimes_{\mathbb{Z}} \mathbb{C}$ through the marking. With a dot denoting the cup-product pairing, define

$$\Phi_m: B \longrightarrow \mathbb{C}^{r(n-1)}, \quad \Phi_m(b) = \left(\gamma_{a,m} \cdot \omega_j(b) \right)_{\substack{1 \leq a \leq r \\ 0 \leq j \leq n-2}}.$$

Then

$$\Phi_m(b_m) = 0, \quad \text{rk}_{\mathbb{C}}(d\Phi_m)_{b_m} = r(n-1).$$

Thus the simultaneous fixed-class Hodge locus $\Phi_m^{-1}(0)$ is smooth of complex codimension $r(n-1)$ near b_m . The moduli stack of Jacobian elliptic surfaces in $\mathcal{W}_n^{\text{sb}}$ equipped with r ordered non-zero sections is smooth at $(S_m, P_{1,m}, \dots, P_{r,m})$, of complex dimension $10n-2-r(n-1)$. After base change to B , forgetting the additional sections identifies its germ at the chosen lift over b_m with $(\Phi_m^{-1}(0), b_m)$.

Consequently, every analytic neighborhood of every point of $\mathcal{W}_{\min,n}$ contains infinitely many pairwise non-isomorphic surfaces in $\mathcal{W}_n^{\text{sb}}$ carrying r independent sections whose canonical heights all exceed any prescribed bound. In particular, the locus of surfaces in $\mathcal{W}_n^{\text{sb}}$ with Mordell–Weil rank at least r is analytically dense in $\mathcal{W}_{\min,n}$.

At a simply-branched surface of height n , the $10n - 2$ ramification points supply independent deformation directions, while each class contributes $n - 1$ period equations. Transverse realization of r classes therefore requires

$$r(n - 1) \leq 10n - 2 \iff r \leq 10 + \left\lfloor \frac{8}{n - 1} \right\rfloor.$$

Equivalently, this bound requires the expected dimension $10n - 2 - r(n - 1)$ of the simultaneous fixed-class Hodge locus to be nonnegative. Thus the transverse range stabilizes at $r \leq 10$ for $n \geq 10$. Beyond this range, the differentials are necessarily dependent and any realization is therefore non-transverse.

The proof uses Shepherd–Barron’s infinitesimal period calculation. Each Schiffer deformation direction indexed by a ramification point induces a rank-one map from $H^{2,0}(S)$ to $H_{\text{prim}}^{1,1}(S)$ [SB23, Thm. 1.4]. We call non-zero generators of these image lines the *Schiffer image classes*; they form an orthogonal basis of primitive $(1, 1)$ -cohomology [SB23, Thm. 4.11(4)]. Evaluation of holomorphic two-forms at distinct ramification points gives Vandermonde minors. Our block construction uses these to show that a maximal Jacobian minor of the simultaneous period equations is not identically zero.

We choose a real tuple in $H_{\text{prim}}^{1,1}(S, \mathbb{R})$ with negative-definite Gram matrix at which the chosen minor does not vanish. We approximate its directions by those of individually primitive lattice vectors whose norms tend to infinity in a fixed auxiliary Euclidean norm. Applying the holomorphic implicit function theorem, with the normalized lattice vectors as parameters, gives nearby surfaces on which all their period pairings vanish. Linearity gives the same vanishing for the original integral vectors, and reality ensures that these vectors have type $(1, 1)$.

Independence persists under sufficiently close approximation. Moreover, the normalized limiting vectors have negative square, so $-\gamma_{a,m}^2 \rightarrow +\infty$ for every a . The Lefschetz $(1, 1)$ -theorem makes these classes algebraic, and on the all- I_1 locus Shioda’s identification turns them into independent sections satisfying

$$\widehat{h}(P_{a,m}) = -\frac{1}{2}\gamma_{a,m}^2 \longrightarrow +\infty.$$

These heights are integers because Λ_n is even.

To obtain exact rank one, suppose that a non-zero class $\alpha \in \Lambda_n$ is indivisible, is of type $(1, 1)$ at a point of $\mathcal{W}_n^{\text{sb}}$, and pairs non-trivially with every Schiffer image class. For each $\beta \in \Lambda_n \setminus \mathbb{Q}\alpha$ that is Hodge at that point, some first-order deformation preserving the Hodge type of α does not preserve that of β . Consequently, outside a countable union of proper closed analytic subsets, a sufficiently small connected representative of the one-class Hodge germ parametrizes surfaces of Mordell–Weil rank exactly one; see Proposition 3.14. Combining this criterion with rational approximation and local realization gives the following consequence, proved in Corollary 3.15.

Corollary 1.3. *For every $n \geq 2$, the locus of surfaces in $\mathcal{W}_n^{\text{sb}}$ with Mordell–Weil rank exactly one is analytically dense in $\mathcal{W}_{\text{min},n}$.*

The one-class argument makes classical infinitesimal Hodge-theoretic criteria explicit in branch coordinates; see Subsection 3.6.

The dimensions of rank-jump loci in moduli of elliptic surfaces have been studied by Cox and Kloosterman. For $n \geq 3$, Cox proves that the locus of height- n Jacobian elliptic surfaces with positive Mordell–Weil rank has dimension $9n - 1$ [Cox90]. Kloosterman determines the dimensions of higher Noether–Lefschetz loci on the nonconstant- j locus and obtains dimension bounds for Mordell–Weil rank loci [Klo07, Thm. 1.1 and Cor. 1.3]. These results provide the dimensional context for our study of analytic density and transverse realization in $\mathcal{W}_n^{\text{sb}}$.

These results fit into the general study of density and distribution of Hodge loci. Khelifa–Urbanik establish density criteria under expected-dimension and monodromy hypotheses [KU25, Thm. 1.9]. Under suitable hypotheses on a weight-two variation, Tayou–Tholozan prove equidistribution of transverse refined Noether–Lefschetz points, ordered by discriminant, when the expected dimension is zero [TT23, Thm. 6.10]. Our proof gives a direct local construction through branch deformations, primitive lattice approximation, and the implicit function theorem. It produces ordered tuples of independent sections whose canonical heights tend to infinity. We also identify the corresponding local moduli of surfaces with sections: after base change to a marked local deformation chart, the forgetful morphism identifies the germ at the chosen lift of the moduli stack of surfaces in $\mathcal{W}_n^{\text{sb}}$ equipped with r ordered non-zero sections with the simultaneous fixed-class Hodge-locus germ.

Remark 1.4. Higher Mordell–Weil ranks occur in classical Delsarte examples with cyclic symmetry; see [Hei12] for a classification and rank computations. Shioda’s isotrivial example has rank 68 at height 60 [Shi92, pp. 109–110]. The non-isotrivial example studied by Shioda [Shi86] and Stiller [Sti87, Thm. 4.2] has rank 56 at height 210. The former has only type-II singular fibres. The latter admits the model

$$y^2 = x^3 + x + t^{1260},$$

whose singular fibres are all of type I_1 , but whose stack-theoretic ramification divisor is

$$\text{Ram}(\varphi_S) = 1259[0] + 839[\infty].$$

Thus neither example lies in $\mathcal{W}_n^{\text{sb}}$. At the same time, Theorem 1.2 implies that every analytic neighbourhood of either example, taken in its corresponding space $\mathcal{W}_{\min,n}$, contains infinitely many pairwise non-isomorphic surfaces in $\mathcal{W}_n^{\text{sb}}$ with Mordell–Weil rank at least 10.

The natural question of whether Mordell–Weil ranks of positive-height Jacobian elliptic surfaces over $\mathbb{P}_{\mathbb{C}}^1$ are unbounded as the height grows remains open [SS10, Question 14.3]. Over each fixed finite field $\mathbb{F}_q$, Mordell–Weil ranks over $\mathbb{F}_q(t)$ are unbounded: Tate–Shafarevich established this for isotrivial curves in odd characteristic [Tv67], and Ulmer constructed non-isotrivial examples in every positive characteristic [Ulm02, Thm. 1.5].

1.1. Outline of the paper. Section 2 records the primitive lattice and the period conditions used in the proof. Section 3 establishes simultaneous transverse realization, constructs independent sections whose canonical heights tend to infinity, and proves analytic density of the stated rank loci, including exact rank one. Subsection 3.7 identifies canonical heights with two-pointed degrees and proves the section-moduli smoothness and germ comparison asserted in Theorem 1.2.

2. PRIMITIVE COHOMOLOGY AND PERIOD CONDITIONS

We recall the lattice and Hodge-theoretic facts used in the realization argument. Our general reference for Hodge theory, period domains, and variations of Hodge structure throughout the paper is Carlson–Müller–Stach–Peters [CMSP17]. We use a dot for the intersection pairing and its complex-bilinear extension.

2.1. The primitive lattice. Let S be a smooth complex projective surface, and let $\pi: S \rightarrow \mathbb{P}^1$ be a relatively minimal Jacobian elliptic fibration with $\chi(\mathcal{O}_S) = n \geq 2$. Let F and O denote a fibre divisor and the zero section, respectively; we use the same symbols for their divisor classes. The canonical-bundle formula gives $K_S = (n-2)F$. Since $F \cdot F = 0$ and $F \cdot O = 1$, adjunction gives $O^2 = -n$. The standard invariant formulas

$$q(S) = 0, \quad p_g(S) = n - 1, \quad e(S) = 12n$$

give $b_2(S) = 12n - 2$; see [Mir89, SS10]. Hodge decomposition and the Hodge index theorem give intersection signature $(2n - 1, 10n - 1)$ on $H^2(S, \mathbb{Z})/\text{tors}$.

The fibre and zero-section classes have intersection matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & -n \end{pmatrix}.$$

Its determinant is -1 , so $\langle F, O \rangle$ is a primitive unimodular sublattice of signature $(1, 1)$. Its orthogonal complement

$$\Lambda_n = \langle F, O \rangle^\perp \subset H^2(S, \mathbb{Z})/\text{tors}$$

therefore has rank $12n - 4$ and signature $(2n - 2, 10n - 2)$. Orthogonal projection gives an integral decomposition

$$H^2(S, \mathbb{Z})/\text{tors} = \langle F, O \rangle \oplus \Lambda_n.$$

We recall the standard description of the primitive lattice and its proof [Gre20, Thm. 24]. The lattice Λ_n is even. Indeed, the Wu formula gives

$$\gamma^2 \equiv \gamma \cdot K_S = (n-2)\gamma \cdot F = 0 \pmod{2} \quad (\gamma \in \Lambda_n).$$

Poincaré duality makes $H^2(S, \mathbb{Z})/\text{tors}$ unimodular, so the orthogonal decomposition also shows that Λ_n is unimodular. The classification of indefinite even unimodular lattices gives

$$(2) \quad \Lambda_n \simeq U^{\oplus(2n-2)} \oplus E_8(-1)^{\oplus n}.$$

Here U is the even hyperbolic plane, with Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and $E_8(-1)$ is the negative-definite E_8 lattice; see [Ser73, Ch. V, Thm. 5]. We use Λ_n also for a fixed abstract model of this lattice.

The primitive Hodge numbers are

$$h^{2,0} = h^{0,2} = n - 1, \quad h_{\text{prim}}^{1,1} = 10n - 2.$$

The intersection pairing is positive definite on the real subspace

$$(H^{2,0}(S) \oplus H^{0,2}(S)) \cap \Lambda_{n, \mathbb{R}}$$

and negative definite on $H_{\text{prim}}^{1,1}(S, \mathbb{R})$. In particular, every non-zero class $\gamma \in \Lambda_n \cap H^{1,1}(S)$ satisfies

$$-\frac{1}{2}\gamma^2 \in \mathbb{Z}_{>0}.$$

Here primitive cohomology means the orthogonal complement of F and O . A primitive lattice vector means a non-zero indivisible vector; these are distinct uses of the word.

2.2. Periods and fixed-class Hodge loci. A primitive Hodge structure is determined by its space of holomorphic two-forms. Let $\mathcal{D}_n^+$ be a connected component of the period domain parametrizing $(n-1)$ -dimensional subspaces $F^2 \subset \Lambda_{n,\mathbb{C}}$ satisfying

$$F^2 \cdot F^2 = 0, \quad \omega \cdot \overline{\omega} > 0 \quad \text{for every } 0 \neq \omega \in F^2.$$

Its complex dimension is

$$\dim_{\mathbb{C}} \mathcal{D}_n^+ = (n-1)(10n-2) + \binom{n-1}{2}.$$

It is a type-IV Hermitian symmetric domain when $n=2$; for $n \geq 3$, it is a Griffiths period domain that is not Hermitian symmetric. For this period-domain description in the elliptic-surface setting, see [Gre20, §3, p. 2345]; for the general Hodge-theoretic background, see [CMSP17, Ch. 4].

For a real class $\gamma \in \Lambda_{n,\mathbb{R}}$, the condition

$$\gamma \perp F^2$$

is equivalent to being of type $(1,1)$. Indeed, reality also gives orthogonality to $\overline{F^2}$, and the orthogonal complement of $F^2 \oplus \overline{F^2}$ is the primitive $(1,1)$ -subspace.

If $\gamma^2 < 0$, its fixed-class Hodge locus in $\mathcal{D}_n^+$ is smooth of complex codimension $n-1$; see [Gre20, §3, p. 2345] and [Gar18, §2.4]. Requiring $F^2 \subset \gamma^\perp$ removes one negative direction from the intersection space, replacing $10n-2$ by $10n-3$ in the dimension formula above. More generally, a fixed negative-definite span of dimension r defines a smooth ambient Hodge locus of complex codimension $r(n-1)$.

To express these conditions on a family, let B be a small simply-connected chart carrying a smooth projective family of height- n Jacobian elliptic surfaces with zero section. The fibre and zero-section classes remain flat and algebraic. Choose a marking at a base point $b_0 \in B$, that is, an isometry from the primitive cohomology lattice of S_{b_0} to Λ_n . Since B is simply-connected, parallel transport extends this to compatible identifications of the primitive cohomology lattices of all fibres with Λ_n . Choose a local holomorphic frame $\omega_0(b), \dots, \omega_{n-2}(b)$ of the Hodge bundle F^2 . For fixed classes $\gamma_1, \dots, \gamma_r \in \Lambda_{n,\mathbb{R}}$, their simultaneous Hodge locus is defined by the holomorphic equations

$$\gamma_a \cdot \omega_j(b) = 0, \quad 1 \leq a \leq r, \quad 0 \leq j \leq n-2.$$

These are $r(n-1)$ equations, but their differentials need not be independent on an elliptic surface deformation space. Section 3 constructs suitable integral classes and nearby surfaces at which all these differentials are independent.

3. GEOMETRIC REALIZATION AND DENSITY OF MORDELL–WEIL RANK JUMPS

We now construct intersections with the elliptic surface period image and identify the resulting integral Hodge classes with Mordell–Weil sections. Fix a height $n \geq 2$, and put

$$h := n - 1, \quad N := 10n - 2.$$

The all- I_1 locus $\mathcal{W}_n^{I_1}$ is obtained from a non-empty Zariski-open subset of the coefficient space

$$H^0(\mathbb{P}^1, \mathcal{O}(4n)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(6n))$$

by quotienting by Weierstrass scaling and changes of coordinates on the base [PS25, Thms. 1.1 and 1.2(a)]. It is therefore irreducible, with

$$\dim \mathcal{W}_n^{I_1} = (4n + 1) + (6n + 1) - 1 - 3 = 10n - 2.$$

We use the primitive lattice and period equations of Section 2. The construction applies to every positive integer r satisfying $rh \leq N$; neither the integral classes nor their Gram matrices are prescribed in advance.

3.1. The simply-branched locus. We first identify a dense open locus on which each ramification point supplies one deformation parameter. These parameters will give the branch coordinates used in the realization argument. Let

$$\varphi_S: \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4, 6)$$

be the classifying morphism of $[S] \in \mathcal{W}_n^{I_1}$; see [HP19, Prop. 9]. Throughout this section, $\text{Ram}(\varphi_S)$ means the ramification divisor of this morphism to the stack $\overline{\mathcal{M}}_{1,1}$, namely the zero divisor of its stack-theoretic differential

$$d\varphi_S: T_{\mathbb{P}^1} \longrightarrow \varphi_S^* T_{\overline{\mathcal{M}}_{1,1}}.$$

It is not the ramification divisor of the coarse j -map; the two notions differ at the orbifold values $j = 0$ and $j = 1728$.

The all- I_1 condition implies that φ_S is non-constant and unramified over the boundary of $\overline{\mathcal{M}}_{1,1}$; see the local calculation in the proof of [HP19, Prop. 11]. Since

$$T_{\overline{\mathcal{M}}_{1,1}} \simeq \mathcal{O}(10), \quad \varphi_S^* \mathcal{O}(1) \simeq \mathcal{O}_{\mathbb{P}^1}(n),$$

the differential is a non-zero section of

$$\varphi_S^* T_{\overline{\mathcal{M}}_{1,1}} \otimes \Omega_{\mathbb{P}^1}^1 \simeq \mathcal{O}_{\mathbb{P}^1}(10n - 2).$$

Consequently,

$$\deg \text{Ram}(\varphi_S) = 10n - 2 = N.$$

Concretely, for Weierstrass data

$$a_4 \in H^0(\mathbb{P}^1, \mathcal{O}(4n)), \quad a_6 \in H^0(\mathbb{P}^1, \mathcal{O}(6n)),$$

the ramification divisor is locally cut out by the weighted Wronskian

$$\Theta(a_4, a_6) := 3a_6 da_4 - 2a_4 da_6.$$

Indeed, the squarefree-discriminant condition excludes a common zero of a_4 and a_6 . The differential is represented by (da_4, da_6) modulo the infinitesimal scaling direction $(4a_4, 6a_6)$. Taking the determinant with the infinitesimal scaling vector represents $d\varphi_S$ by $4a_4 da_6 - 6a_6 da_4 = -2\Theta(a_4, a_6)$. Hence $\Theta(a_4, a_6)$ cuts out $\text{Ram}(\varphi_S)$, including multiplicities.

Under a change of local frame of $\mathcal{O}(n)$ by a unit u , the coefficient functions become $u^{-4}a_4$ and $u^{-6}a_6$. The terms involving du cancel, giving

$$\Theta(u^{-4}a_4, u^{-6}a_6) = u^{-10}\Theta(a_4, a_6).$$

Thus these local expressions define a global section

$$\Theta(a_4, a_6) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(10n) \otimes \Omega_{\mathbb{P}^1}^1) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(10n-2)).$$

Definition 3.1. A surface $[S] \in \mathcal{W}_n^{\text{I}_1}$ is *simply-branched* if the stack-theoretic ramification divisor $\text{Ram}(\varphi_S)$ is reduced. We denote the corresponding substack by

$$\mathcal{W}_n^{\text{sb}} \subset \mathcal{W}_n^{\text{I}_1}.$$

Thus, after passing to an étale neighbourhood which orders the ramification divisor, a point of $\mathcal{W}_n^{\text{sb}}$ has

$$\text{Ram}(\varphi_S) = p_1 + \cdots + p_N$$

with the p_i pairwise distinct. This definition does not require the images $\varphi_S(p_i)$ to be pairwise distinct.

Lemma 3.2. *The substack $\mathcal{W}_n^{\text{sb}}$ is Zariski open and dense in $\mathcal{W}_n^{\text{I}_1}$.*

Proof. Reducedness of the zero divisor of $\Theta(a_4, a_6)$ is a Zariski-open condition on the Weierstrass coefficients. Thus $\mathcal{W}_n^{\text{sb}}$ is Zariski open in $\mathcal{W}_n^{\text{I}_1}$. To prove density, we show that multiple ramification points can be split by deforming the surface.

We use Shepherd–Barron’s local deformation theory. The height- n , base-genus-zero part of his stack $\mathcal{J}\mathcal{E}^{\text{gen}}$ is our all- I_1 locus [SB23, p. 2523]. Its classifying morphisms are non-constant and unramified over the boundary; their ramification divisors need not be reduced. His numerical hypotheses hold because the base has genus zero and $p_g = n-1 > 0$. Moreover, the relatively-minimal surface coincides with its minimal Weierstrass model, since there are no fibre components disjoint from the zero section.

Here $\text{Def}(S)$ denotes the miniversal analytic deformation space of the Jacobian elliptic fibration with its zero section, with the base considered up to isomorphism. It underlies a local quotient chart of $\mathcal{W}_n^{\text{I}_1}$ at $[S]$ [SB23, Prop. 2.9(2)–(3) and Prop. 4.4(1)].

Suppose that the ramification divisor of a given surface has multiplicity $e_i - 1$ at a point p_i . Shepherd–Barron’s local isomorphism [SB23, Prop. 4.4] identifies this miniversal chart with the product of the deformation spaces of the finite schemes

$$\text{Spec } \mathbb{C}[t]/(t^{e_i}).$$

The i th local deformation base has dimension $e_i - 1$. Since

$$\sum_i (e_i - 1) = \deg \text{Ram}(\varphi_S) = N,$$

their product is an N -dimensional miniversal chart. On the simply-branched locus, each factor is one-dimensional, giving one deformation parameter for each ramification point. More generally, the product description allows us to deform the ramification at the different points independently. To split a point of multiplicity $e - 1$, lift φ_S locally to an étale chart of the target stack and choose coordinates w

on the source and s on that chart. Shepherd–Barron’s Schiffer deformation has the form [SB23, Prop. 4.3]

$$s = w^e + e \sum_{j=0}^{e-2} t_{e-j} w^j.$$

Up to the non-zero factor e , its critical divisor is therefore cut out by

$$w^{e-1} + \sum_{j=1}^{e-2} j t_{e-j} w^{j-1}.$$

For $e = 2$ this divisor is already reduced. For $e \geq 3$, setting $t_{e-1} \neq 0$ and all other relevant parameters equal to zero gives

$$w^{e-1} + t_{e-1},$$

which has distinct roots. Thus the discriminant of the local critical polynomial is not identically zero. Choose pairwise disjoint coordinate discs around the points p_i . By Rouché’s theorem, after shrinking the deformation chart each disc contains exactly $e_i - 1$ critical points, counted with multiplicity. These account for the total ramification degree N , so there are no critical points outside their union. The simultaneous non-vanishing of the local discriminants defines a dense open subset of the miniversal chart. Thus every surface in $\mathcal{W}_n^{l_1}$ is an analytic limit of surfaces with reduced ramification.

Since $\mathcal{W}_n^{l_1}$ is irreducible and $\mathcal{W}_n^{\text{sb}}$ is a non-empty Zariski-open substack, $\mathcal{W}_n^{\text{sb}}$ is Zariski dense in $\mathcal{W}_n^{l_1}$. ■

Lemma 3.3. *For every $n \geq 2$, both inclusions*

$$\mathcal{W}_n^{\text{sb}} \subset \mathcal{W}_n^{l_1} \subset \mathcal{W}_{\min, n}$$

are Zariski open and dense. In particular, $\mathcal{W}_n^{\text{sb}}$ is analytically dense in $\mathcal{W}_{\min, n}$.

Proof. Consider the irreducible coefficient space

$$V_n = H^0(\mathbb{P}^1, \mathcal{O}(4n)) \oplus H^0(\mathbb{P}^1, \mathcal{O}(6n)).$$

Minimality is a Zariski-open condition on the Weierstrass data [BPS24, Prop. 4.15]. Intersecting with the open condition $\Delta \neq 0$ gives a non-empty Zariski-open subset of V_n presenting $\mathcal{W}_{\min, n}$ after quotienting by Weierstrass scaling and changes of coordinates on the base.

Reducedness of the discriminant divisor on $\mathbb{P}^1$ is also a Zariski-open condition. This condition is non-empty: in homogeneous coordinates $[s : t]$ on $\mathbb{P}^1$, take

$$a_4 = -3s^{4n}, \quad a_6 = t^{6n}.$$

Then

$$\Delta = -16(4a_4^3 + 27a_6^2) = -432(t^{12n} - 4s^{12n})$$

has $12n$ distinct zeros, and it does not vanish at $[s : t] = [0 : 1]$. These Weierstrass data are minimal. Indeed, non-minimality at a point would force the discriminant to vanish there to order at least 12.

Thus the squarefree-discriminant locus is a non-empty Zariski-open subset of the irreducible minimal elliptic coefficient locus. Passing to the quotient shows that $\mathcal{W}_n^{l_1}$ is Zariski open and dense in $\mathcal{W}_{\min, n}$.

Lemma 3.2 gives the same conclusion for $\mathcal{W}_n^{\text{sb}} \subset \mathcal{W}_n^{\text{I}_1}$. Hence $\mathcal{W}_n^{\text{sb}}$ is Zariski open and dense in $\mathcal{W}_{\min, n}$. A non-empty Zariski-open subset of an irreducible complex variety is analytically dense; applying this on the coefficient presentation gives the analytic assertion. $\blacksquare$

3.2. The Shepherd–Barron frame. Fix $[S] \in \mathcal{W}_n^{\text{sb}}$ and order its ramification points as $p_1, \dots, p_N$. Choose an affine coordinate z on the base for which none of the p_i lies at infinity, and put $z_i = z(p_i)$. On the corresponding affine Weierstrass chart, write

$$y^2 = x^3 + a_4(z)x + a_6(z), \quad \omega_0 = \frac{dx \wedge dz}{2y}.$$

The canonical bundle formula identifies global holomorphic two-forms with polynomials in z of degree at most $n - 2$, multiplied by ω_0 . Thus

$$H^{2,0}(S) = \bigoplus_{j=0}^{h-1} \mathbb{C} \omega_j, \quad \omega_j = z^j \omega_0.$$

The form ω_0 does not vanish along any finite fibre. We use this polynomial identification for sections of $\mathcal{O}_{\mathbb{P}^1}(n - 2)$ below; in particular, $f(p_i)$ means the scalar $f(z_i)$.

Every automorphism of the Jacobian fibration preserves its zero section and the reduced discriminant divisor of degree $12n$. Its induced automorphism of $\mathbb{P}^1$ therefore belongs to the finite group preserving these $12n \geq 3$ points, while the kernel embeds in the finite group $\text{Aut}_{\mathbb{C}(t)}(E_S, 0)$. Thus

$$G := \text{Aut}_{\mathcal{W}_n^{\text{sb}}}([S])$$

is finite. Since $\mathcal{W}_n^{\text{sb}}$ is an open substack of $\mathcal{W}_{\min, n}$, it is smooth and separated Deligne–Mumford by [PS25, Thm. 1.2(a)–(b)]. By the analytic local quotient theorem [ES18, Prop. 2.9], we may choose a sufficiently small simply-connected analytic ball B , a point $0 \in B$ lifting $[S]$, and an analytic quotient chart

$$[B/G] \longrightarrow \mathcal{W}_n^{\text{sb}}.$$

After shrinking B , we may assume that the induced map from B/G identifies it with an analytic neighbourhood of the image of $[S]$ in the coarse moduli space.

The relative ramification divisor is finite étale of degree N over B . Since B is simply-connected, the ordering at 0 extends to an ordering throughout B . We trivialize the primitive integral local system over B by flat transport. By the preceding deformation-space identification and [SB23, §4, Prop. 4.4], after shrinking B there are analytic coordinates

$$(t_1, \dots, t_N)$$

such that $\partial/\partial t_i|_0$ spans the Schiffer direction associated with p_i . Shepherd–Barron associates to each p_i a meromorphic two-form $\tilde{\eta}_i$ of the second kind, meaning that its residue along the fibre over p_i vanishes. Its class is taken in the primitive quotient $F^\perp/\mathbb{C}F$, identified with $\Lambda_{n, \mathbb{C}}$ using the zero section. Taking its $(1, 1)$ -component gives

$$\eta_i = \text{pr}^{1,1}[\tilde{\eta}_i] \in H_{\text{prim}}^{1,1}(S).$$

Since the ramification divisor is reduced, the classes $\eta_1, \dots, \eta_N$ form an orthogonal basis of $H_{\text{prim}}^{1,1}(S)$ for the intersection pairing [SB23, Thm. 4.11(4)].

Write $E_{p_i} = \pi^{-1}(p_i)$ for the fibre over p_i . Let F^2 denote the holomorphic Hodge bundle over B , whose fibre at b is $H^{2,0}(S_b)$. The Gauss–Manin connection ∇ differentiates sections of the cohomology bundle using its flat trivialization. Since

$$K_S \simeq \pi^* \mathcal{O}_{\mathbb{P}^1}(n-2),$$

the canonical linear system is base-point free for $n \geq 2$. Shepherd–Barron’s rank-one formula therefore applies [SB23, Thm. 1.4(3)]. For every $\omega \in H^{2,0}(S)$, it gives

$$(\nabla_{\partial/\partial t_i} \omega)^{1,1} = c_i(\omega) \eta_i.$$

The derivative is evaluated at $0 \in B$ using any local holomorphic extension of ω in F^2 . Its $(1,1)$ -projection is independent of that extension: changing the extension only adds an element of $H^{2,0}(S)$ to the derivative; see [CMSP17, Lem. 5.3.2]. Here c_i is a non-zero linear functional with

$$\ker(c_i) = H^0(S, \Omega_S^2(-E_{p_i})).$$

Griffiths transversality places the derivative in $H^{2,0}(S) \oplus H_{\text{prim}}^{1,1}(S)$; Shepherd–Barron’s formula further shows that, for each Schiffer direction, the $(1,1)$ -components of the derivatives of all holomorphic two-forms lie in the same line $\mathbb{C}\eta_i$. The kernel description identifies c_i with evaluation at p_i , up to a non-zero scalar, giving the following formula.

Lemma 3.4. *For $f \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2))$ one has*

$$c_i(f \omega_0) = c_i(\omega_0) f(p_i), \quad c_i(\omega_0) \neq 0.$$

In particular,

$$c_i(\omega_j) = c_i(\omega_0) z_i^j, \quad 0 \leq j \leq h-1.$$

Proof. The functional $f \mapsto c_i(f \omega_0)$ is non-zero and has kernel

$$H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(n-2)(-p_i)).$$

The evaluation functional $f \mapsto f(p_i)$ has the same kernel. The two functionals are therefore proportional, and evaluation at $f = 1$ gives the stated constant. $\blacksquare$

The evaluation formula lets us test whether the h period equations for a fixed class have independent differentials. The following criterion does this by selecting h ramification points.

Theorem 3.5. *Let γ be a locally flat real class which is of type $(1,1)$ at $[S] \in \mathcal{W}_n^{\text{sb}}$. If there are distinct indices $i_1, \dots, i_h$ such that*

$$\gamma \cdot \eta_{i_k} \neq 0 \quad (1 \leq k \leq h),$$

then, on the marked local deformation space, the fixed-class type- $(1,1)$ locus defined by γ is smooth of codimension $h = n-1$ at $[S]$.

Proof. Extend $\omega_0, \dots, \omega_{h-1}$ to a holomorphic frame of F^2 near $[S]$ and set

$$f_j = \gamma \cdot \omega_j.$$

For a real flat class, vanishing of these pairings is equivalent to being of type $(1, 1)$; the conjugate equations follow automatically. Since γ is flat and of type $(1, 1)$ at $[S]$, Lemma 3.4 gives

$$\left. \frac{\partial f_j}{\partial t_{i_k}} \right|_{[S]} = c_{i_k}(\omega_0) z_{i_k}^j (\gamma \cdot \eta_{i_k}).$$

If J_γ denotes this $h \times h$ minor, then

$$\det J_\gamma = \left(\prod_{k=1}^h c_{i_k}(\omega_0) (\gamma \cdot \eta_{i_k}) \right) \prod_{1 \leq a < b \leq h} (z_{i_b} - z_{i_a}).$$

The first product is non-zero by Lemma 3.4 and the hypothesis on γ . The second is non-zero because the ramification points are distinct. Hence the Jacobian of $(f_0, \dots, f_{h-1})$ has rank h at $[S]$. These equations define the fixed-class type- $(1, 1)$ locus, so the holomorphic implicit function theorem makes this locus smooth of codimension h at $[S]$. ■

The same period calculation shows that every non-zero fixed integral class in Λ_n has a proper Hodge locus on a marked local chart. Together with the algebraicity of Hodge loci, this gives the following.

Proposition 3.6. *For every $n \geq 2$, a very general member of $\mathcal{W}_{\min, n}$ has Mordell–Weil rank zero.*

Proof. At any point of $\mathcal{W}_n^{\text{sb}}$, every non-zero integral Hodge class in Λ_n pairs non-trivially with some Schiffer image class. The rank-one formula therefore gives a deformation direction in which one of its periods has non-zero derivative. Consequently, every non-zero fixed-class Hodge locus on a marked local chart is a proper closed analytic subset. Since the lattice is countable, Baire’s theorem shows that their union has empty analytic interior.

Apply the algebraicity theorem of Cattani–Deligne–Kaplan [CDK95] to the primitive variation on a smooth algebraic atlas of finite type for $\mathcal{W}_n^{\text{sb}}$. The locus carrying a non-zero integral Hodge class is a countable union of closed algebraic subsets of this atlas. Their images in $\mathcal{W}_n^{\text{sb}}$ are constructible. None is dense, since a dense constructible image would contain a non-empty open subset, contradicting the local empty-interior assertion.

Their closures in $\mathcal{W}_{\min, n}$, together with the proper closed complement of $\mathcal{W}_n^{\text{sb}}$, form a countable union of proper closed substacks. Outside this union there are no non-zero integral Hodge classes in Λ_n . Since all singular fibres are irreducible, the Shioda–Tate formula gives Mordell–Weil rank zero. ■

3.3. Simultaneous period equations. For r classes, we need rh period equations with independent differentials. We first construct a complex tuple for which the Jacobian splits into r Vandermonde blocks, then use polynomial non-vanishing to obtain a real tuple.

Lemma 3.7. *Let $r \geq 1$, put $M = rh$, and suppose $M \leq N$. There exist $\mathbb{R}$ -linearly independent vectors*

$$v_1, \dots, v_r \in W := H_{\text{prim}}^{1,1}(S, \mathbb{R}), \quad v_a^2 = -1,$$

and a subset $I \subset \{1, \dots, N\}$ of cardinality M such that the $M \times M$ matrix

$$J_{(a,j),i} = c_i(\omega_0) z_i^j (v_a \cdot \eta_i), \quad 1 \leq a \leq r, \quad 0 \leq j \leq h-1, \quad i \in I,$$

is non-singular.

Proof. Choose any M indices and partition them into disjoint blocks

$$I = C_1 \sqcup \dots \sqcup C_r, \quad |C_a| = h.$$

Over $\mathbb{C}$, set

$$v_a^\circ = \sum_{i \in C_a} \frac{\eta_i}{\eta_i^2}.$$

The denominators are non-zero because the η_i form an orthogonal basis of a non-degenerate quadratic space. Orthogonality gives

$$v_a^\circ \cdot \eta_i = \begin{cases} 1, & i \in C_a, \\ 0, & i \notin C_a. \end{cases}$$

After ordering rows and columns by the blocks, the resulting matrix is block diagonal. Its a th block has determinant

$$\left(\prod_{i \in C_a} c_i(\omega_0) \right) \prod_{\substack{i, k \in C_a \\ i < k}} (z_k - z_i),$$

which is non-zero.

Thus the determinant is a non-zero polynomial on

$$(H_{\text{prim}}^{1,1}(S))^r = (W \otimes_{\mathbb{R}} \mathbb{C})^r.$$

A complex polynomial which vanishes on the real form W^r vanishes identically. Hence we may choose a real tuple on which the determinant is non-zero. The chosen vectors are linearly independent: any relation $\sum_a \lambda_a v_a = 0$ would give the same relation among the rows indexed by $(a, 0)$, contradicting determinant non-vanishing. Finally, negative definiteness allows us to replace each v_a by $v_a / \sqrt{-v_a^2}$. This multiplies its h corresponding rows by the same non-zero scalar, so the determinant remains non-zero. $\blacksquare$

To pass from the real tuple to integral classes, we approximate its directions by primitive lattice vectors of increasing norm. These approximations may leave the original Hodge subspace; we will subsequently deform the surface to make the chosen integral classes Hodge.

Lemma 3.8. *Let $L \subset E$ be a full lattice, meaning a discrete subgroup spanning the real vector space E . Assume $\dim E \geq 2$ and equip E with a Euclidean norm. Let $v_1, \dots, v_r \in E$ be linearly independent unit vectors. There exist primitive vectors*

$$\gamma_{a,m} \in L, \quad 1 \leq a \leq r,$$

such that

$$\|\gamma_{a,m}\| \longrightarrow \infty, \quad \frac{\gamma_{a,m}}{\|\gamma_{a,m}\|} \longrightarrow v_a,$$

and, for all sufficiently large m , the vectors $\gamma_{1,m}, \dots, \gamma_{r,m}$ are linearly independent. If a proper real subspace $E_0 \subsetneq E$ contains all the v_a , the vectors $\gamma_{a,m}$ may moreover be chosen outside E_0 .

Proof. After choosing a basis of L , the directions of primitive lattice vectors are precisely the rational directions, which are dense in the unit sphere of E . For each m , only finitely many primitive vectors have norm at most m ; deleting their finitely many directions does not destroy density. If $E_0 \subsetneq E$ is a proper real subspace, rational directions outside E_0 are still dense: every open subset of the unit sphere contains a smaller open subset disjoint from E_0 , and rational directions are dense there. For each a and m , we may therefore choose a primitive vector $\gamma_{a,m}$, outside E_0 when specified, such that

$$\|\gamma_{a,m}\| > m, \quad \left\| \frac{\gamma_{a,m}}{\|\gamma_{a,m}\|} - v_a \right\| < \frac{1}{m}.$$

These inequalities give the two asserted limits. Since linear independence is an open condition on tuples of vectors, convergence to the independent tuple $(v_1, \dots, v_r)$ gives independence for all sufficiently large m . ■

3.4. Local simultaneous realization. The implicit function theorem lets us compensate for a small change in the cohomology classes by a small deformation of the surface. We apply this to primitive lattice vectors whose normalized directions approach the real tuple constructed above, while their norms tend to infinity.

Theorem 3.9. *Let $n \geq 2$, let $[S] \in \mathcal{W}_n^{\text{sb}}$, and let $r \geq 1$ satisfy*

$$r(n-1) \leq 10n-2.$$

Then there are points representing pairwise non-isomorphic Jacobian elliptic surfaces,

$$[S_m] \in \mathcal{W}_n^{\text{sb}}, \quad [S_m] \longrightarrow [S],$$

and, after identifying the primitive cohomology lattices by flat transport, primitive classes

$$\gamma_{1,m}, \dots, \gamma_{r,m} \in \Lambda_n$$

with the following properties:

- (1) *every $\gamma_{a,m}$ is of type $(1, 1)$ on S_m ;*
- (2) *the classes $\gamma_{1,m}, \dots, \gamma_{r,m}$ span a negative-definite subspace and are linearly independent;*
- (3) *on the marked local deformation space, the $r(n-1)$ period equations defining their simultaneous fixed-class Hodge locus have linearly independent differentials at $[S_m]$; in particular, this locus is smooth of complex codimension $r(n-1)$ there;*
- (4) *for every a ,*

$$d_{a,m} := -\frac{\gamma_{a,m}^2}{2} \in \mathbb{Z}_{>0}, \quad d_{a,m} \longrightarrow \infty.$$

Proof. Put $M = rh$. Work on the simply-connected chart B above. Flat transport identifies the real primitive cohomology of its fibres with $\Lambda_{n,\mathbb{R}}$. At the central surface, Hodge decomposition gives an orthogonal splitting

$$\Lambda_{n,\mathbb{R}} = H_+ \oplus W,$$

where

$$H_+ = (H^{2,0}(S) \oplus H^{0,2}(S)) \cap \Lambda_{n,\mathbb{R}}, \quad W = H_{\text{prim}}^{1,1}(S, \mathbb{R}).$$

The intersection form is positive definite on H_+ and negative definite on W . We therefore use the positive-definite norm determined by

$$\|v + w\|^2 = v^2 - w^2, \quad v \in H_+, \quad w \in W.$$

In particular, the vectors supplied by Lemma 3.7 have norm one.

Choose $v_1, \dots, v_r \in W$ and $I \subset \{1, \dots, N\}$ as in Lemma 3.7. Let $s = (t_i)_{i \in I}$ be the corresponding M coordinates, and let $B_I \subset B$ be the slice on which all other coordinates are zero. Restrict B_I to a sufficiently small coordinate polydisc $\mathbb{D}^M \subset B$ with coordinates s , where $\mathbb{D} \subset \mathbb{C}$ is an open disc centred at zero. Choose a holomorphic frame

$$\omega_0(t), \dots, \omega_{h-1}(t)$$

of F^2 over B whose value at the origin is the monomial basis above. With the notation $\omega_j(s) = \omega_j(t)|_{B_I}$, define the restriction of the Hodge-equation map to this slice

$$\Psi: (\Lambda_{n,\mathbb{C}})^r \times \mathbb{D}^M \longrightarrow \mathbb{C}^{rh}$$

by

$$\Psi_{a,j}(\mathbf{u}, s) = u_a \cdot \omega_j(s), \quad 1 \leq a \leq r, \quad 0 \leq j \leq h-1.$$

Because $v_a \in W$, one has $\Psi(\mathbf{v}, 0) = 0$. Moreover, Lemma 3.4 gives

$$\left. \frac{\partial \Psi_{a,j}}{\partial t_i} \right|_{(\mathbf{v}, 0)} = c_i(\omega_0) z_i^j (v_a \cdot \eta_i),$$

so the derivative with respect to s is non-singular by Lemma 3.7.

The holomorphic implicit function theorem therefore gives a neighbourhood $\mathcal{V}$ of $\mathbf{v} = (v_1, \dots, v_r)$ in $(\Lambda_{n,\mathbb{C}})^r$ and a holomorphic map

$$\tau: \mathcal{V} \longrightarrow \mathbb{D}^M, \quad \tau(\mathbf{v}) = 0,$$

such that

$$(3) \quad \Psi(\mathbf{u}, \tau(\mathbf{u})) = 0$$

for every $\mathbf{u} \in \mathcal{V}$. Shrinking $\mathcal{V}$ if necessary, the derivative with respect to s remains non-singular along the graph of τ .

The period equations are homogeneous in each class, so normalization does not change their zero locus. Since $h > 0$, the positive subspace H_+ is non-zero and $W \subset \Lambda_{n,\mathbb{R}}$ is proper. We may therefore apply Lemma 3.8 to the lattice $\Lambda_n \subset \Lambda_{n,\mathbb{R}}$ with $E_0 = W$. Choose primitive vectors $\gamma_{a,m} \notin W$ such that

$$u_{a,m} := \frac{\gamma_{a,m}}{\|\gamma_{a,m}\|} \longrightarrow v_a, \quad \|\gamma_{a,m}\| \longrightarrow \infty.$$

For large m , the tuple $\mathbf{u}_m$ lies in $\mathcal{V}$. Let $[S_m]$ be the point of the chart with selected coordinates $\tau(\mathbf{u}_m)$ and all remaining coordinates zero. Then $[S_m] \rightarrow [S]$, and (3) yields

$$\gamma_{a,m} \cdot \omega = 0 \quad \text{for every } \omega \in F^2(S_m) \quad \text{and every } 1 \leq a \leq r.$$

The classes $\gamma_{a,m}$ are real, so they are also orthogonal to $\overline{F^2(S_m)}$ and hence are of type $(1, 1)$.

For the fixed tuple $(\gamma_{1,m}, \dots, \gamma_{r,m})$, the simultaneous Hodge locus in the full chart B is defined by the rh equations

$$\gamma_{a,m} \cdot \omega_j(t) = 0, \quad 1 \leq a \leq r, \quad 0 \leq j \leq h-1.$$

Their Jacobian contains the selected $M \times M$ minor from the implicit function argument. Replacing $u_{a,m}$ by $\gamma_{a,m}$ multiplies each of the h rows belonging to a by $\|\gamma_{a,m}\|$, so this minor remains non-singular. The full Jacobian therefore has rank $M = rh$, and the simultaneous fixed-class Hodge locus is smooth of codimension M at $[S_m]$.

The normalized intersection Gram matrices converge:

$$\frac{\gamma_{a,m} \cdot \gamma_{b,m}}{\|\gamma_{a,m}\| \|\gamma_{b,m}\|} \longrightarrow v_a \cdot v_b.$$

The limiting matrix is negative definite because the v_a are linearly independent vectors in W . Hence the classes $\gamma_{1,m}, \dots, \gamma_{r,m}$ span a negative-definite subspace for all sufficiently large m . Since $v_a^2 = -1$, one has

$$\frac{d_{a,m}}{\|\gamma_{a,m}\|^2} = -\frac{\gamma_{a,m}^2}{2\|\gamma_{a,m}\|^2} \longrightarrow \frac{1}{2}.$$

Therefore $d_{a,m} \rightarrow \infty$. Each $d_{a,m}$ is an integer because Λ_n is even, and it is positive for all sufficiently large m .

Finally, $\gamma_{a,m} \notin W$ implies that it is not of type $(1, 1)$ at the central surface, so $\tau(u_m) \neq 0$. These non-zero points tend to zero; after passing to a subsequence they are pairwise distinct. Since G is finite, a further subsequence lies in pairwise distinct G -orbits. By the choice of the uniformizing chart, their images in the coarse analytic moduli space are pairwise distinct; equivalently, the corresponding Jacobian elliptic surfaces are pairwise non-isomorphic. All the resulting points lie in $\mathcal{W}_n^{\text{sb}}$ by construction. After discarding finitely many initial terms and relabelling the final subsequence, properties (1)–(4) hold for every m . ■

3.5. Mordell–Weil rank, heights, and analytic density. Since every fibre is irreducible, the Shioda–Tate description removes only the fibre and zero-section classes. The primitive integral Hodge classes constructed above therefore give Mordell–Weil sections, and their intersection pairing determines the canonical heights. We then use density of the simply-branched locus to obtain statements throughout the minimal elliptic locus.

Theorem 3.10. *Let $n \geq 2$ and*

$$1 \leq r \leq \left\lfloor \frac{10n-2}{n-1} \right\rfloor.$$

For every $[S] \in \mathcal{W}_n^{\text{sb}}$ there are pairwise non-isomorphic Jacobian elliptic surfaces

$$[S_m] \in \mathcal{W}_n^{\text{sb}}, \quad [S_m] \longrightarrow [S],$$

and linearly independent sections

$$P_{1,m}, \dots, P_{r,m} \in \text{MW}(S_m/\mathbb{P}^1)$$

such that

$$\widehat{h}(P_{a,m}) \longrightarrow \infty \quad (1 \leq a \leq r).$$

In particular,

$$\text{rk MW}(S_m/\mathbb{P}^1) \geq r.$$

The corresponding Shioda classes are primitive in Λ_n , and the canonical heights are positive integers. On the marked local deformation space, the $r(n-1)$ period equations defining their simultaneous fixed-class Hodge locus have linearly independent differentials at $[S_m]$; hence this locus is smooth of complex codimension $r(n-1)$ there.

Proof. Take the surfaces and classes supplied by Theorem 3.9. Since S_m has singular fibres, its Néron–Severi group is torsion-free [Shi90, Thm. 1.2]. We therefore identify $\text{NS}(S_m)$ with its image in $H^2(S_m, \mathbb{Z})/\text{tors}$. The Lefschetz $(1, 1)$ -theorem gives $\gamma_{a,m} \in \text{NS}(S_m)$ for every a [CMSP17, §3.4, pp. 119–120].

Put $U_{S_m} = \langle F, O \rangle$. Since F and O are algebraic, the integral orthogonal splitting from Section 2 restricts to

$$\text{NS}(S_m) = U_{S_m} \oplus (\text{NS}(S_m) \cap \Lambda_n).$$

All fibres are irreducible, so $\text{Triv}(S_m) = U_{S_m}$. The Shioda–Tate description [Shi90, Thm. 1.3] and the preceding decomposition therefore give

$$\text{MW}(S_m/\mathbb{P}^1) \simeq \text{NS}(S_m)/U_{S_m} \simeq \text{NS}(S_m) \cap \Lambda_n.$$

Under this identification, the map from sections to algebraic classes in Λ_n is Shioda’s orthogonal projection σ_{Sh} [Shi90, Lem. 8.1]. Consequently, each $\gamma_{a,m}$ determines a unique section $P_{a,m}$ with

$$\sigma_{\text{Sh}}(P_{a,m}) = \gamma_{a,m}.$$

Since this correspondence is an isomorphism of abelian groups, the linear independence of the classes gives the linear independence of the sections.

Shioda’s height pairing [Shi90, Thm. 8.4], with our half-normalization, gives

$$\widehat{h}(P_{a,m}) = -\frac{1}{2} \sigma_{\text{Sh}}(P_{a,m})^2 = -\frac{1}{2} \gamma_{a,m}^2 = d_{a,m} \in \mathbb{Z}_{>0}, \quad d_{a,m} \longrightarrow \infty.$$

The Shioda classes are exactly the classes constructed in Theorem 3.9, so their primitivity and the smoothness and codimension of their simultaneous Hodge locus follow from that theorem. $\blacksquare$

Corollary 3.11. *Let $n \geq 2$. Every analytic neighbourhood of every point of $\mathcal{W}_{\min,n}$ contains a sequence of pairwise non-isomorphic Jacobian elliptic surfaces $[S_m] \in \mathcal{W}_n^{\text{sb}}$ with sections $P_m \in \text{MW}(S_m/\mathbb{P}^1)$ whose canonical heights are strictly increasing positive integers. In particular, infinitely many positive integers occur as section heights within every such neighbourhood.*

Proof. Let U be an analytic neighbourhood of any point of $\mathcal{W}_{\min,n}$. By Lemma 3.3, choose $[S] \in U \cap \mathcal{W}_n^{\text{sb}}$. Apply Theorem 3.10 at $[S]$ with $r = 1$. The resulting pairwise non-isomorphic surfaces converge to $[S]$, so all sufficiently late members lie in U . Their section heights are positive integers tending to infinity. Passing to a subsequence makes these heights strictly increasing. $\blacksquare$

Corollary 3.12. *Let $n \geq 2$ and*

$$1 \leq r \leq \left\lfloor \frac{10n-2}{n-1} \right\rfloor.$$

Then

$$\{[S] \in \mathcal{W}_n^{\text{sb}} : \text{rk MW}(S/\mathbb{P}^1) \geq r\}$$

is dense in $\mathcal{W}_{\min,n}$ in the analytic topology.

More precisely, let $[S_0]$ be any point of $\mathcal{W}_{\min,n}$, let U be any analytic neighbourhood of $[S_0]$, and let $H > 0$. Then U contains infinitely many surfaces $[S'] \in \mathcal{W}_n^{\text{sb}}$, pairwise non-isomorphic as Jacobian elliptic surfaces, carrying independent sections

$$P_1, \dots, P_r \in \text{MW}(S'/\mathbb{P}^1)$$

such that

$$\widehat{h}(P_a) > H \quad (1 \leq a \leq r).$$

The corresponding Shioda classes may be chosen primitive. The $r(n-1)$ period equations defining their simultaneous fixed-class Hodge locus have linearly independent differentials on a marked local deformation space at $[S']$; hence this locus is smooth of complex codimension $r(n-1)$ there.

Proof. By Lemma 3.3, choose $[S] \in U \cap \mathcal{W}_n^{\text{sb}}$. Apply Theorem 3.10 at $[S]$ to obtain pairwise non-isomorphic surfaces $[S_m] \rightarrow [S]$ carrying r independent sections $P_{a,m}$. Since r is fixed and each sequence of heights tends to infinity,

$$\min_{1 \leq a \leq r} \widehat{h}(P_{a,m}) \longrightarrow \infty.$$

Thus, for all sufficiently large m , the surface $[S_m]$ lies in U and every selected section has height greater than H . Primitivity of the Shioda classes and the smoothness and codimension of their simultaneous Hodge locus follow from the same theorem. Removing finitely many terms leaves infinitely many pairwise non-isomorphic surfaces, proving the assertion. $\blacksquare$

3.6. Exact Mordell–Weil rank one. The one-class realization argument can be sharpened to give exact Mordell–Weil rank one outside a countable union of proper closed analytic subsets of a suitable Hodge locus. The underlying infinitesimal argument is classical: compare conditions (1) and (3) and the proof of [Voi07, Thm. 3.1]. The density step is a local form of Green’s density criterion, recalled in [Voi07, Prop. 3.3]. Here Shepherd–Barron’s frame makes the required criterion explicit: non-vanishing against every Schiffer image class prevents any independent primitive Hodge direction from persisting along the imposed one-class locus. We give the calculation and its Mordell–Weil consequences in full.

Keep $n \geq 2$, $h = n-1$, and $N = 10n-2$. Work on a sufficiently small full marked deformation chart B in $\mathcal{W}_n^{\text{sb}}$, centred at 0, and write S_b for the surface over $b \in B$. Use the Schiffer tangent basis, the distinct finite ramification coordinates $z_1, \dots, z_N$, and the basis $\eta_1, \dots, \eta_N$ of $H_{\text{prim}}^{1,1}(S_0)$ constructed above. Extend $\omega_0, \dots, \omega_{h-1}$ to a holomorphic frame of F^2 over B . For a flat real class $\gamma \in \Lambda_{n,\mathbb{R}}$, define

$$(4) \quad \Phi_\gamma(b) = (\gamma \cdot \omega_0(b), \dots, \gamma \cdot \omega_{h-1}(b)), \quad Z_\gamma = \Phi_\gamma^{-1}(0).$$

Since γ is real, Z_γ is precisely its type- $(1, 1)$ locus.

If γ is Hodge at 0, Lemma 3.4 identifies its period Jacobian in the Schiffer tangent basis as

$$(5) \quad d\Phi_\gamma(0) = A_\gamma = V \text{diag}(c_i(\omega_0)(\gamma \cdot \eta_i))_{i=1}^N, \quad V = (z_i^j)_{\substack{0 \leq j < h \\ 1 \leq i \leq N}}.$$

We keep the non-zero factors $c_i(\omega_0)$ from the preceding construction.

Lemma 3.13. *Let $z_1, \dots, z_N$ be distinct complex numbers, let $h \geq 1$, and assume $N \geq h$ and $N > 2h - 2$. Set $V = (z_i^j)_{0 \leq j < h, 1 \leq i \leq N}$. For $a \in (\mathbb{C}^*)^N$ and $b \in \mathbb{C}^N$, put*

$$A = V \operatorname{diag}(a), \quad C = V \operatorname{diag}(b).$$

If $\ker A \subseteq \ker C$, then $b = ca$ for some $c \in \mathbb{C}$.

Proof. The first h columns of V have determinant

$$\prod_{1 \leq i < k \leq h} (z_k - z_i) \neq 0,$$

so A has row rank h . Every row of C annihilates $\ker A$, hence belongs to the row space of A . Applying this to the first row gives a polynomial p of degree at most $h - 1$ such that

$$(6) \quad b_i = a_i p(z_i) \quad (1 \leq i \leq N).$$

For $h = 1$, this proves the assertion.

For $h \geq 2$, apply the same argument to the last row of C . There is a polynomial q of degree at most $h - 1$ satisfying

$$z_i^{h-1} b_i = a_i q(z_i) \quad (1 \leq i \leq N).$$

Dividing by $a_i \neq 0$ and using (6), we find that $z^{h-1} p(z) - q(z)$ vanishes at all N points z_i . Its degree is at most $2h - 2 < N$, so it vanishes identically. If $p \neq 0$, the identity $z^{h-1} p = q$ gives

$$h - 1 + \deg p = \deg q \leq h - 1.$$

Thus p is constant. If $p = 0$, then $b = 0$. In either case $b = ca$. ■

Proposition 3.14. *Let $\alpha \in \Lambda_n$ be non-zero, indivisible in Λ_n , and of type $(1, 1)$ on S_0 . Assume*

$$(7) \quad \alpha \cdot \eta_i \neq 0 \quad (1 \leq i \leq N).$$

Then Z_α is smooth of complex codimension $h = n - 1$ at 0. For every $\beta \in \Lambda_n \setminus \mathbb{Q}\alpha$ that is Hodge at 0, one has

$$(8) \quad d\Phi_\beta(0)|_{T_0 Z_\alpha} \neq 0.$$

After replacing Z_α by a sufficiently small connected representative of its germ, there is a countable union $\mathcal{E}$ of proper closed analytic subsets of Z_α such that

$$(9) \quad \Lambda_n \cap H^{1,1}(S_b) = \mathbb{Z}\alpha, \quad \operatorname{rk} \operatorname{MW}(S_b/\mathbb{P}^1) = 1 \quad (b \in Z_\alpha \setminus \mathcal{E}).$$

These points are dense in Z_α and have full measure for every measure absolutely continuous with respect to intrinsic smooth volume on Z_α .

Proof. Condition (7) makes the diagonal factor in A_α invertible. Since V has row rank h , the map $d\Phi_\alpha(0)$ is surjective. The holomorphic implicit function theorem gives smoothness of codimension h and

$$(10) \quad T_0 Z_\alpha = \ker A_\alpha.$$

Shrink the chart so that Z_α is a connected complex manifold.

Let $\beta \in \Lambda_n \setminus \mathbb{Q}\alpha$ be Hodge at 0. If (8) failed, then

$$\ker A_\alpha \subseteq \ker A_\beta.$$

Apply Lemma 3.13 with

$$a_i = c_i(\omega_0)(\alpha \cdot \eta_i), \quad b_i = c_i(\omega_0)(\beta \cdot \eta_i).$$

The numerical hypotheses hold since

$$N \geq h, \quad N - (2h - 2) = 8n + 2 > 0.$$

The lemma gives $b_i = ca_i$ for some $c \in \mathbb{C}$. Cancelling $c_i(\omega_0) \neq 0$, we obtain

$$(\beta - c\alpha) \cdot \eta_i = 0 \quad (1 \leq i \leq N).$$

Both α and β lie in $H_{\text{prim}}^{1,1}(S_0)$. The intersection pairing on this space is non-degenerate: its real form is negative definite by the Hodge index theorem. Since the η_i form a basis, the displayed equalities force $\beta = c\alpha$ in $\Lambda_{n,\mathbb{C}}$. Comparing an integral lattice coordinate in which α is non-zero gives $c \in \mathbb{Q}$, a contradiction. This proves (8).

For each $\beta \in \Lambda_n \setminus \mathbb{Q}\alpha$, the subset $Z_\alpha \cap Z_\beta$ is closed analytic in Z_α . If β is not Hodge at 0, this subset does not contain 0 and is therefore proper. If β is Hodge at 0, equation (8) shows that Φ_β cannot vanish identically on Z_α , so the subset is again proper. Thus

$$\mathcal{E} = \bigcup_{\beta \in \Lambda_n \setminus \mathbb{Q}\alpha} (Z_\alpha \cap Z_\beta)$$

is a countable union of proper closed analytic subsets. Each is nowhere dense and has measure zero with respect to every smooth positive density on Z_α . Baire's theorem gives density of the complement, and countable additivity gives the measure assertion.

At $b \notin \mathcal{E}$, every integral Hodge class in Λ_n belongs to $\mathbb{Q}\alpha$. Since α is indivisible,

$$\Lambda_n \cap \mathbb{Q}\alpha = \mathbb{Z}\alpha.$$

This proves the first equality in (9).

Finally, the integral Shioda identification established in the proof of Theorem 3.10 applies to every surface in $\mathcal{W}_n^{\text{sb}}$. Since all fibres are irreducible and $\langle F, O \rangle$ is unimodular, Lefschetz (1, 1) and the Shioda–Tate description give

$$\text{MW}(S_b/\mathbb{P}^1) \simeq \text{NS}(S_b) \cap \Lambda_n = H^{1,1}(S_b) \cap \Lambda_n.$$

For $b \notin \mathcal{E}$, the last group is $\mathbb{Z}\alpha$. Hence $\text{rk MW}(S_b/\mathbb{P}^1) = 1$, proving the second assertion in (9). $\blacksquare$

Equation (8) describes the geometric mechanism. Among the collective branch deformations that preserve the Hodge type of α , there is a direction that obstructs any specified independent additional Hodge class. The non-vanishing condition therefore controls which global sections can persist while the branch data vary.

Corollary 3.15. *For every $n \geq 2$, the locus*

$$\{[S] \in \mathcal{W}_n^{\text{sb}} : \text{rk MW}(S/\mathbb{P}^1) = 1\}$$

is dense in $\mathcal{W}_{\min,n}$ for the complex analytic topology.

Proof. Let U be an arbitrary non-empty analytic open subset of $\mathcal{W}_{\min,n}$. By Lemma 3.3, choose $[S_0] \in U \cap \mathcal{W}_n^{\text{sb}}$, and take a full marked deformation chart B centred at 0 whose image lies in $U \cap \mathcal{W}_n^{\text{sb}}$.

For each i , the real-linear functional

$$H_{\text{prim}}^{1,1}(S_0, \mathbb{R}) \longrightarrow \mathbb{C}, \quad \lambda \longmapsto \lambda \cdot \eta_i$$

is not identically zero. Otherwise its complex-linear extension would vanish on $H_{\text{prim}}^{1,1}(S_0)$, contradicting non-degeneracy of the pairing and $\eta_i \neq 0$. A finite union of proper real-linear subspaces cannot cover a real vector space. We may therefore choose

$$(11) \quad \lambda \in H_{\text{prim}}^{1,1}(S_0, \mathbb{R}), \quad \lambda \cdot \eta_i \neq 0 \quad (1 \leq i \leq N).$$

Using the marking, consider the jointly holomorphic map

$$(12) \quad \Psi_1 : \Lambda_{n, \mathbb{C}} \times B \longrightarrow \mathbb{C}^h, \quad \Psi_1(v, b) = (v \cdot \omega_j(b))_{0 \leq j < h}.$$

We have $\Psi_1(\lambda, 0) = 0$, and (5) and (11) show that the derivative in the B variables is surjective. Choose h chart coordinates giving an invertible minor and fix the remaining coordinates at their values at 0. The holomorphic implicit function theorem gives a neighbourhood $\mathcal{V}_\lambda$ of λ in $\Lambda_{n, \mathbb{C}}$ and a holomorphic map

$$\tau_1 : \mathcal{V}_\lambda \longrightarrow B, \quad \tau_1(\lambda) = 0, \quad \Psi_1(v, \tau_1(v)) = 0.$$

Choose non-zero rational vectors $v_m \in \Lambda_{n, \mathbb{Q}} \cap \mathcal{V}_\lambda$ converging to λ , and put $b_m = \tau_1(v_m)$. Then $b_m \rightarrow 0$. Each v_m is real and satisfies the period equations, so it is of type $(1, 1)$ on S_{b_m} .

After shrinking B , label the simple ramification points continuously and choose continuous non-zero representatives $\eta_i(b)$ of the corresponding Schiffer image lines in $H_{\text{prim}}^{1,1}(S_b)$, agreeing with η_i at 0. The existence of these continuous representatives follows from the line-bundle description over the universal reduced ramification divisor in [SB23, Prop. 5.1], together with the smooth identification of the Hodge quotient F^1/F^2 with the $(1, 1)$ -subspace. Under the marking,

$$v_m \cdot \eta_i(b_m) \longrightarrow \lambda \cdot \eta_i \neq 0.$$

For all sufficiently large m , every pairing on the left is therefore non-zero.

Let α_m be an indivisible generator of the rank-one lattice $\Lambda_n \cap \mathbb{Q}v_m$. Multiplication by a non-zero rational scalar preserves both the Hodge condition and these non-vanishing pairings. Thus α_m satisfies the hypotheses of Proposition 3.14 at b_m . That proposition supplies exact-rank-one points arbitrarily close to b_m on its one-class Hodge locus. Their images lie in U . Since U was arbitrary, the required density follows. $\blacksquare$

3.7. Two-pointed moduli and deformations of sections. In this subsection, we interpret the realization theorem in terms of moduli of elliptic surfaces with chosen sections. We first express the canonical height of a section as the degree of a line bundle pulled back from $\overline{\mathcal{M}}_{1,2}$. We then compare deformations of the sections with the corresponding period conditions. Inchiostro identifies $\mathcal{X} = \overline{\mathcal{M}}_{1,2}$ with the $(4, 6)$ -weighted blowup of $\mathcal{P}(2, 3, 4)$ at $[1 : 1 : 0]$ [Inc22, Thms. 1.1 and 2.10]. Put $\mathcal{Y} = \overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4, 6)$, and write

$$\phi : \mathcal{X} \longrightarrow \mathcal{Y}, \quad p : \mathcal{X} \longrightarrow \mathcal{P}(2, 3, 4)$$

for the universal stable curve and the weighted blowdown, respectively. All weighted projective spaces here are stacks. The exceptional divisor D parametrizes curves

whose two markings lie on a rational tail; under the universal-curve identification, it is the zero section. We write

$$\lambda = \mathcal{O}_{\mathcal{Y}}(1), \quad \mathcal{L}_{1,2} = \phi^* \lambda, \quad \xi = p^* \mathcal{O}_{\mathcal{P}(2,3,4)}(1).$$

Throughout this subsection, we work on the all- I_1 locus.

Proposition 3.16. *The line bundles on $\mathcal{X}$ satisfy*

$$(13) \quad \xi \simeq \mathcal{L}_{1,2} \otimes \mathcal{O}_{\mathcal{X}}(D), \quad \omega_{\mathcal{X}} \simeq \mathcal{L}_{1,2}^{\otimes(-9)}.$$

Let $S \rightarrow \mathbb{P}^1$ represent a point of $\mathcal{W}_n^{I_1}$, with classifying map φ_S and fundamental line bundle $\mathcal{L}$. A section $P \neq O$ determines a lift $f_P: \mathbb{P}^1 \rightarrow \mathcal{X}$ of φ_S . If $C_P = P^*O$ is the divisor where P meets O , then

$$(14) \quad f_P^* \mathcal{L}_{1,2} \simeq \mathcal{L}, \quad f_P^* D = C_P, \quad (p \circ f_P)^* \mathcal{O}(1) \simeq \mathcal{L}(C_P).$$

In particular,

$$(15) \quad \deg f_P^* \mathcal{L}_{1,2} = n, \quad \widehat{h}(P) = \deg f_P^* \xi = n + (P \cdot O) \geq n.$$

Proof. We first compute the line bundles in (13). In weighted coordinates $[x : y : A]$ on $\mathcal{P}(2, 3, 4)$, the rational map to $\mathcal{Y}$ is given by A and $y^2 - x^3 - Ax$, of weights 4 and 6. Their pullbacks along p have exceptional factors of orders 4 and 6. Dividing by these factors gives the pair defining ϕ , with associated line bundle $\xi \otimes \mathcal{O}_{\mathcal{X}}(-D)$. Thus $\mathcal{L}_{1,2} \simeq \xi \otimes \mathcal{O}_{\mathcal{X}}(-D)$ in the integral Picard group of $\mathcal{X}$.

The canonical bundle of $\mathcal{P}(2, 3, 4)$ is $\mathcal{O}(-9)$, and the $(4, 6)$ -weighted blowup has discrepancy $4 + 6 - 1 = 9$. Hence

$$\omega_{\mathcal{X}} \simeq \xi^{\otimes(-9)} \otimes \mathcal{O}_{\mathcal{X}}(9D) \simeq \mathcal{L}_{1,2}^{\otimes(-9)},$$

which proves the second identity.

We now construct f_P . Since all singular fibres have type I_1 , the surface S is the pullback of the universal stable one-pointed curve along φ_S . The section P therefore determines a lift to $\mathcal{X}$. A section cannot pass through a fibre node: in the local equation $xy = t$, the conditions $x(0) = y(0) = 0$ would imply $\text{ord}_0(xy) \geq 2$.

At a point where P meets O , the two-pointed stabilization inserts a rational tail carrying the two markings. Since D is the universal zero section, its pullback is P^*O as an effective Cartier divisor, with the collision multiplicities included. Together with $\varphi_S^* \lambda \simeq \mathcal{L}$ and the first identity in (13), this proves (14). The map f_P lands in $\mathcal{M}_{1,2}$ precisely where the fibre is smooth and the two sections are distinct.

It remains to identify the degree with the canonical height. Put $d = n + (P \cdot O)$. The class

$$\gamma_P = [P] - [O] - d[F]$$

is orthogonal to F because $(P - O) \cdot F = 0$ and $F^2 = 0$. The coefficient d is forced by orthogonality to O , since $(P - O) \cdot O = n + P \cdot O = d$ and $F \cdot O = 1$. Since $P^2 = O^2 = -n$ and $P \cdot F = O \cdot F = 1$, we have $\gamma_P^2 = -2d$. All fibres are irreducible, so γ_P is Shioda's projection of P . Our normalization gives $\widehat{h}(P) = -\gamma_P^2/2 = d$, while (14) gives $\deg f_P^* \xi = d$. Finally, $P \neq O$ implies $P \cdot O \geq 0$, proving (15). $\blacksquare$

We can also describe the map to $\mathcal{P}(2, 3, 4)$ directly from the Weierstrass equation. Write $P = (x_p, y_p)$ in $y^2 = x^3 + Ax + B$, let s be the canonical section of $\mathcal{O}(C_p)$, and put $M = \mathcal{L}(C_p)$. Then

$$X = s^2 x_p \in H^0(M^{\otimes 2}), \quad Y = s^3 y_p \in H^0(M^{\otimes 3}), \quad A' = s^4 A \in H^0(M^{\otimes 4})$$

satisfy $Y^2 - X^3 - A'X = s^6 B$ and define $p \circ f_p = [X : Y : A']$. To check that these sections have no common zero, consider first a collision of multiplicity k . The coordinates x_p and y_p have poles of orders $2k$ and $3k$, so X and Y are nonzero there and the image is $[1 : 1 : 0]$. Away from the collisions, a common zero of X, Y, A' would imply $A = B = 0$, contrary to the all- I_1 hypothesis. Thus the associated line bundle is $M = \mathcal{L}(C_p)$.

We next introduce the moduli stack of surfaces with several chosen sections. For an ordered tuple $\mathbf{d} = (d_1, \dots, d_r)$ of positive integers, let $\mathcal{S}_{n, \mathbf{d}}^{\text{I}_1}$ denote the stack of height- n all- I_1 Jacobian elliptic surfaces with an ordered tuple of nonzero sections $P_1, \dots, P_r$ satisfying $\widehat{h}(P_a) = d_a$. The base is considered up to isomorphism. The sections determine a map to the fibre product

$$\mathcal{X}^{(r)} = \underbrace{\mathcal{X} \times_{\mathcal{Y}} \cdots \times_{\mathcal{Y}} \mathcal{X}}_{r \text{ factors}}.$$

Conversely, a map $f = (f_1, \dots, f_r): \mathbb{P}^1 \rightarrow \mathcal{X}^{(r)}$ has a common projection $\varphi: \mathbb{P}^1 \rightarrow \mathcal{Y}$. Pulling back the universal curve along φ gives the elliptic fibration, and the lifts f_a give its sections. We require $\deg \varphi^* \lambda = n$, $\deg f_a^* \xi = d_a$, and the pullback of the nodal boundary to be reduced. The positivity of d_a excludes the zero section, since $\xi|_D \simeq \mathcal{O}_D$. These constructions commute with base change. Thus $\mathcal{S}_{n, \mathbf{d}}^{\text{I}_1}$ is the corresponding open locus of maps of the specified degrees, modulo PGL_2 . The fibre product allows the sections P_a to meet.

Let $\rho: \mathcal{S}_{n, \mathbf{d}}^{\text{I}_1} \rightarrow \mathcal{W}_n^{\text{I}_1}$ be the morphism forgetting the additional sections. Its source and target have finite stabilizers; automorphisms are required to preserve the zero section. The degrees $d_a = \deg f_a^* \xi$ are locally constant in families, so fixing $\mathbf{d}$ does not change the local deformation germ of a tuple.

Proposition 3.17. *Let $(S, P_1, \dots, P_r)$ be a point of $\mathcal{S}_{n, \mathbf{d}}^{\text{I}_1}$ and let $B \rightarrow \mathcal{W}_n^{\text{I}_1}$ be a simply-connected analytic étale local deformation chart through $[S]$. Choose a marking and a lift of the tuple over the central point b_0 . Put $\gamma_a = [P_a] - [O] - d_a[F]$, and define*

$$\Phi(b) = (\gamma_a \cdot \omega_j(b))_{a, j}, \quad 1 \leq a \leq r, \quad 0 \leq j \leq n-2,$$

where $\omega_0(b), \dots, \omega_{n-2}(b)$ is a local holomorphic frame of $H^{2,0}(S_b)$. If $d\Phi$ has rank $r(n-1)$ at b_0 , then $\mathcal{S}_{n, \mathbf{d}}^{\text{I}_1}$ is smooth at the tuple, of dimension

$$(16) \quad 10n - 2 - r(n-1).$$

After base change to B , the morphism ρ identifies the germ at the chosen lift with $(\Phi^{-1}(0), b_0)$.

Proof. We first compute the deformations of the corresponding map $f = (f_{P_1}, \dots, f_{P_r}): \mathbb{P}^1 \rightarrow \mathcal{X}^{(r)}$. Each component lies in the smooth locus of ϕ . This includes collisions with O , which correspond to smooth points of the original one-pointed fibre under the

universal-curve identification. Thus $\mathcal{X}^{(r)}$ is smooth in a neighbourhood of $f(\mathbb{P}^1)$. Since

$$N_{P_a/S} \simeq \mathcal{O}_{\mathbb{P}^1}(-n), \quad \varphi_S^* T_Y \simeq \mathcal{L}^{\otimes 10} \simeq \mathcal{O}_{\mathbb{P}^1}(10n),$$

the relative tangent sequence gives, with $E = f^* T_{\mathcal{X}^{(r)}}$,

$$(17) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-n)^{\oplus r} \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^1}(10n) \longrightarrow 0.$$

Taking cohomology gives

$$(18) \quad 0 \longrightarrow H^0(E) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^1}(10n)) \xrightarrow{\delta} \bigoplus_{a=1}^r H^1(N_{P_a/S}) \longrightarrow H^1(E) \longrightarrow 0.$$

The component δ_a assigns to a first-order deformation of φ_S the obstruction to lifting the section P_a .

To pass to deformations over an unparameterized base, we quotient by PGL_2 . This gives

$$T_{b_0} B \simeq H^0(\varphi_S^* T_Y) / d\varphi_S(H^0(T_{\mathbb{P}^1})).$$

The denominator has dimension three, since an infinitesimal source stabilizer would preserve the $12n$ distinct discriminant points and hence vanish. These reparametrizations lift through df , so δ descends to $T_{b_0} B$.

We next compare this obstruction map with the period derivative. For a section $P = P_a$, consider

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_S(P) \longrightarrow N_{P/S} \longrightarrow 0.$$

Since $H^0(N_{P/S}) = 0$, we have $h^0(\mathcal{O}_S(P)) = 1$. Also $H^2(\mathcal{O}_S(P)) = 0$ by Serre duality: the divisor $K_S - P$ has intersection -1 with the nef fibre class, so it cannot be effective. Riemann–Roch gives

$$\chi(\mathcal{O}_S(P)) = n + \frac{1}{2}P \cdot (P - K_S) = n + \frac{1}{2}(-n - (n - 2)) = 1.$$

It follows that $H^1(\mathcal{O}_S(P)) = 0$. The connecting map, also called the divisor semiregularity map, is therefore an isomorphism

$$(19) \quad s_P: H^1(N_{P/S}) \xrightarrow{\sim} H^2(\mathcal{O}_S).$$

For the Cartier divisor $P \subset S$, the semiregularity map of Buchweitz–Flenner agrees with s_P : both are Serre dual to the restriction map $H^0(K_S) \rightarrow H^0(K_S|_P)$. Compatibility with the Kodaira–Spencer map identifies the image of the section obstruction under s_P with the infinitesimal Hodge obstruction for $[P]$; see [BF00, Thms. 1.3, 4.5 and 5.5(1)–(2), §5.3]. Since $[O]$ and $[F]$ remain Hodge, this is also the obstruction for γ_a . Applying the maps s_{P_a} and Serre duality $H^2(\mathcal{O}_S) \simeq H^0(K_S)^\vee$, then evaluating in the chosen frame, identifies the descended δ with $d\Phi$, up to the sign convention for the connecting maps.

The rank hypothesis now implies that δ is surjective. By (18), $H^1(E) = 0$. For maps from the fixed curve $\mathbb{P}^1$ to a smooth Deligne–Mumford stack, the tangent space is $H^0(E)$ and an obstruction space is $H^1(E)$. Indeed, local lifts across a square-zero extension differ by sections of E , and their obstruction to gluing is the resulting Čech class in $H^1(E)$. Thus the mapping stack is smooth at f . Using (17), its dimension is

$$h^0(E) = \chi(E) = (10n + 1) + r(1 - n).$$

Quotienting by PGL_2 subtracts 3, proving (16). In particular, for $r = 1$ under the rank hypothesis, the actual dimension equals the expected dimension $9n - 1$, independently of d_1 .

It remains to identify the germs. The morphism ρ is representable: for a fixed surface family, the tuples of sections form an open locus in a product of relative Hilbert spaces. Hence the germ H of $\mathcal{S}_{n,d}^{I_1} \times_{\mathcal{W}_n^{I_1}} B$ at the chosen lift is an analytic space, smooth by the preceding argument.

The divisors P_a preserve their marked integral classes in this family. Thus the period functions Φ vanish on H , and, since H is reduced, ρ factors holomorphically through $\Phi^{-1}(0)$. The tangent sequence identifies TH with $\ker(d\Phi)$; there are no additional tangent directions from sections on a fixed surface because $H^0(N_{P_a/S}) = 0$. Both germs are smooth of dimension (16). The analytic inverse function theorem therefore identifies them. Smoothness here is asserted for the incidence stack and its local charts. ■

The sections constructed in Theorem 3.10 satisfy the transversality hypothesis of the preceding proposition. We therefore obtain the following moduli-theoretic interpretation of the realization theorem.

Corollary 3.18. *Let $n \geq 2$ and $r \geq 1$ be integers satisfying $r(n-1) \leq 10n-2$. For every $[S] \in \mathcal{W}_n^{\mathrm{sb}}$, the surfaces and individually primitive sections of Theorem 3.10 give points*

$$(S_m, P_{1,m}, \dots, P_{r,m}) \in \mathcal{S}_{n,d_m}^{I_1}, \quad d_{a,m} = \widehat{h}(P_{a,m}) \longrightarrow \infty.$$

At each such point, the stack is smooth of dimension $10n-2-r(n-1)$, and its germ on the chosen marked chart is the simultaneous fixed-class Hodge-locus germ. Moreover,

$$\deg f_{P_{a,m}}^* \mathcal{L}_{1,2} = n, \quad \deg(p \circ f_{P_{a,m}})^* \mathcal{O}(1) = d_{a,m}, \quad (P_{a,m} \cdot O_m) = d_{a,m} - n.$$

Every analytic neighbourhood of every point of $\mathcal{W}_{\min,n}$ contains infinitely many pairwise non-isomorphic surfaces with these smooth section-moduli germs and with the heights of all chosen sections exceeding any prescribed bound.

Proof. Theorem 3.10 supplies the primitive independent sections with heights tending to infinity and full-rank period derivatives. Proposition 3.17 gives the smoothness and the identification of germs, while Proposition 3.16 gives the degree formulas.

For the last assertion, choose a simply-branched surface in the given neighbourhood using Lemma 3.3, and apply the realization theorem at that surface. All sufficiently late members of the resulting sequence lie in the neighbourhood and satisfy the height bounds. ■

ACKNOWLEDGEMENTS

The author is indebted to Michael Atiyah for an inspiring discussion and for his encouragement to explore connections between arithmetic and physics. The author also thanks Dori Bejleri for discussions of modularity and Mordell–Weil rank jumps, François Greer and Klaus Hulek for careful comments on an earlier draft, and Scott Mullane and Matthew Satriano for helpful conversations. The

author was partially supported by the ARC grant DP210103397 and the Sydney Mathematical Research Institute.

REFERENCES

- [BF00] Ragnar-Olaf Buchweitz and Hubert Flenner. The Atiyah–Chern character yields the semiregularity map as well as the infinitesimal Abel–Jacobi map. In *The Arithmetic and Geometry of Algebraic Cycles*, volume 24 of *CRM Proceedings & Lecture Notes*, pages 33–46. American Mathematical Society, Providence, RI, 2000.
- [BHPVdV04] Wolf P Barth, Klaus Hulek, Chris A. M. Peters, and Antonius Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, second edition, 2004.
- [BPS24] Dori Bejleri, Jun-Yong Park, and Matthew Satriano. Height moduli on cyclotomic stacks and counting elliptic curves over function fields, 2024. arXiv:2210.04450v2.
- [CDK95] Eduardo Cattani, Pierre Deligne, and Aroldo Kaplan. On the locus of Hodge classes. *J. Amer. Math. Soc.*, 8(2):483–506, 1995.
- [CMSP17] James Carlson, Stefan Müller-Stach, and Chris Peters. *Period Mappings and Period Domains*, volume 168 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2 edition, 2017.
- [Cox90] David A. Cox. The Noether–Lefschetz locus of regular elliptic surfaces with section and $p_g \geq 2$. *Amer. J. Math.*, 112(2):289–329, 1990.
- [ES18] Philippe Eyssidieux and Francesco Sala. Instantons and framed sheaves on Kähler Deligne–Mumford stacks. *Ann. Fac. Sci. Toulouse Math. (6)*, 27(3):599–628, 2018.
- [Gar18] Luis E. Garcia. Superconnections, theta series, and period domains. *Adv. Math.*, 329:555–589, 2018.
- [Gre20] François Greer. Modular forms from Noether–Lefschetz theory. *Algebra Number Theory*, 14(9):2335–2368, 2020.
- [Hei12] Bas Heijne. The maximal rank of elliptic Delsarte surfaces. *Math. Comp.*, 81(278):1111–1130, 2012.
- [HP19] Changho Han and Jun-Yong Park. Arithmetic of the moduli of semistable elliptic surfaces. *Math. Ann.*, 375(3-4):1745–1760, 2019.
- [Huy16] D. Huybrechts. *Lectures on K3 Surfaces*, volume 158 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2016.
- [Inc22] Giovanni Inchiostro. Moduli of genus one curves with two marked points as a weighted blow-up. *Math. Z.*, 302(3):1905–1925, 2022.
- [Klo07] Remke Kloosterman. Higher Noether–Lefschetz loci of elliptic surfaces. *J. Differential Geom.*, 76(2):293–316, 2007.
- [KU25] Nazim Khelifa and David Urbanik. Existence and density of typical Hodge loci. *Math. Ann.*, 391(1):819–842, 2025.
- [Mir89] Rick Miranda. *The basic theory of elliptic surfaces*. Dottorato di Ricerca in Matematica. [Doctorate in Mathematical Research]. ETS Editrice, Pisa, 1989.
- [Ogu03] Keiji Oguiso. Local families of K3 surfaces and applications. *J. Algebraic Geom.*, 12(3):405–433, 2003.
- [PS25] Jun-Yong Park and Johannes Schmitt. Arithmetic geometry of the moduli stack of Weierstrass fibrations over $\mathbb{P}^1$. *Math. Z.*, 310(4):Paper No. 84, 33, 2025.
- [SB23] N. I. Shepherd-Barron. Generic Torelli and local Schottky theorems for Jacobian elliptic surfaces. *Compos. Math.*, 159(12):2521–2550, 2023.
- [Ser73] J.-P. Serre. *A course in arithmetic*, volume No. 7 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Heidelberg, 1973. Translated from the French.
- [Shi86] Tetsuji Shioda. An explicit algorithm for computing the Picard number of certain algebraic surfaces. *Amer. J. Math.*, 108(2):415–432, 1986.

- [Shi90] Tetsuji Shioda. On the Mordell-Weil lattices. *Comment. Math. Univ. St. Paul.*, 39(2):211–240, 1990.
- [Shi92] Tetsuji Shioda. Some remarks on elliptic curves over function fields. *Astérisque*, 209:99–114, 1992. Journées Arithmétiques, 1991 (Geneva).
- [SS10] Matthias Schütt and Tetsuji Shioda. Elliptic surfaces. In *Alg. Geom. in East Asia—Seoul 2008*, volume 60 of *Adv. Stud. Pure Math.*, pages 51–160. Math. Soc. Japan, 2010.
- [Sti87] Peter F. Stiller. The Picard numbers of elliptic surfaces with many symmetries. *Pacific J. Math.*, 128(1):157–189, 1987.
- [TT23] Salim Tayou and Nicolas Tholozan. Equidistribution of Hodge loci II. *Compos. Math.*, 159(1):1–52, 2023.
- [Tv67] John Tate and I. R. Šafarevič. The rank of elliptic curves. *Dokl. Akad. Nauk SSSR*, 175:770–773, 1967.
- [Ulm02] Douglas Ulmer. Elliptic curves with large rank over function fields. *Ann. of Math. (2)*, 155(1):295–315, 2002.
- [Voi07] Claire Voisin. Hodge loci and absolute Hodge classes. *Compositio Mathematica*, 143(4):945–958, 2007.

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