Height moduli spaces and exact Šafarevič counts for elliptic curves over function fields

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Rational points on projective varieties over $\mathbb{Q}$

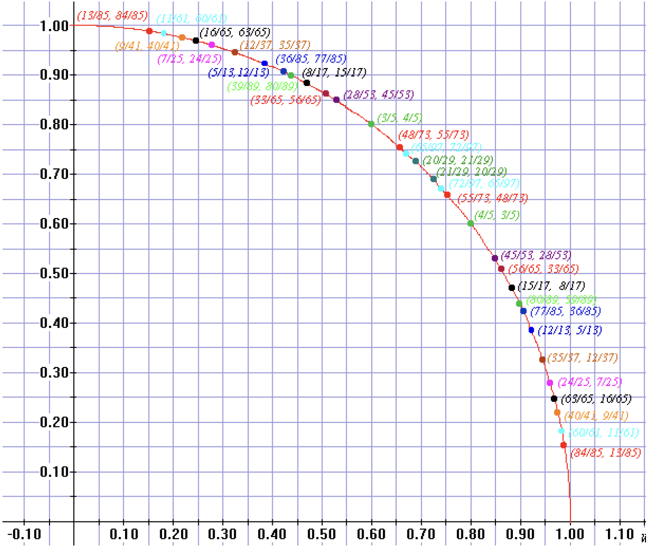


Figure 1: Rational points on the unit circle $x^2 + y^2 = 1$ over $\mathbb{Q}$

1. Geometry is enlightening and quadratic formula is awesome

$$(x, y) = \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right) \in \mathbb{Q}^2$$

we parametrized all rational points on the unit circle over $\mathbb{Q}$

2. Integral points on the projective conic

$$C := V(X^2 + Y^2 - Z^2) \subset \mathbb{P}^2$$

are precisely the *Pythagorean triples*:

$$[X : Y : Z] = [a^2 - b^2 : 2ab : a^2 + b^2] \in \mathbb{Z}^3$$

3. Height measures arithmetic complexity: for $\gcd(a, b) = 1$,

$$\mathrm{ht}\left(\frac{a}{b}\right) = \max(|a|, |b|)$$

$\mathrm{ht}(4/10) = 5$ and
$\mathrm{ht}(1000000001/1000000000) = 1000000001 \neq 1$

4. On **projective varieties**, the integral and the rational points coincide (viewing $[a : b]$ with $\gcd(a, b) = 1$)

$$X(\mathbb{Q}) = X(\mathbb{Z})$$

1. Parametrization needs a starting rational point. For instance, the conic $x^2 + y^2 = 3$ has *no* rational point: $X(\mathbb{Q}) = \emptyset$. Showing this is genuinely *arithmetic* - Fermat's method of infinite descent or by Hasse–Minkowski for conics it suffices to find one place v with $X(\mathbb{Q}_v) = \emptyset$; here $X(\mathbb{Q}_3) = \emptyset$.

2. Higher degree curves are "Fermat's Last Theorem". For $x^4 + y^4 = 1$ (a genus-3 curve), one has by *Wiles-Taylor*

$$X(\mathbb{Q}) = \{(\pm 1, 0), (0, \pm 1)\}$$

More generally, by the *Mordell–Faltings theorem*, if $X/\mathbb{Q}$ is a smooth projective curve of genus ≥ 2, then $X(\mathbb{Q})$ is finite.

3. For an elliptic curve over $\mathbb{Q}$ in Weierstrass form

$$E: \; y^2 = x^3 + Ax + B,$$

there is still no general algorithm that, given $A, B \in \mathbb{Q}$, always determines the full set $E(\mathbb{Q})$ explicitly.

4. But we do have a canonical rational point: the point at infinity

$$\infty = [0 : 1 : 0] \in E(\mathbb{Q}), \qquad E = V\big(Y^2 Z - X^3 - AXZ^2 - BZ^3\big) \subset \mathbb{P}^2$$

Degree of countable infinity, the rank

(1) (**Mordell–Weil**) For an elliptic curve $E/\mathbb{Q}$, the group of rational points is finitely generated:

$$E(\mathbb{Q}) \;\cong\; \mathbb{Z}^r \;\oplus\; E(\mathbb{Q})_{\mathrm{tors}},$$

where $r \geq 0$ is the Mordell–Weil rank and $E(\mathbb{Q})_{\mathrm{tors}}$ is finite.

(2) Fundamental questions about r:

- An algorithm that is guaranteed to correctly compute r?
- Which values of r can occur? *How often do they occur?*
- Is there an upper limit? *Can r be arbitrarily large?*

(3) Records: the current best explicit examples with largest rank is Elkies–Klagsbrun (announced 2024) produced an $E/\mathbb{Q}$ with *rank at least* 29 (the previous unconditional record was ≥ 28 from Elkies, 2006).

Trivial torsion group, rank ≥ 29

Elkies - Klagsbrun (2024)

$$y^2 + xy = x^3 - 27006183241630922218434652145297453784768054621836357954737385x$$
$$+ 5525805855134237647573669959111819182152106703253507960837240477914941327771617342563672149 7$$

Independent points of infinite order:

P_{1} = [28911954742285371894582555366 34, 115993074809612470645983591072731867959342528 3]

P_{2} = [3402542165322127811451484642234, 1661508223164691055862657623730465560755290883]

P_{3} = [4298760026558467240422107564794, 4313142249890236204790986787384907722927474563]

P_{4} = [3728756667770947009884455714554, 2530180219584734091116528693531660545660397443]

P_{5} = [5991744132052078230511185130234, 10418901628842034362301273055728300669218858883]

P_{6} = [3236493534632768520540227223034, 1324626796262167243658687198416201825373745283]

P_{7} = [78226686134991174232380689386234, 69039421006275989650342965412551677999951255488 3]

P_{8} = [11492605643548859374635605140234, 35536316911450952155461624238308456029618940883]

P_{9} = [-5143303362384229804906088118566, 7622356511107986864120352355674305680222368483]

P_{10} = [4439856555750654352815684350 02, 6584468124388858623214803939643557365635620355]

P_{11} = [-9795650189042696807526297497 66, 8987348422104537684966706438714038633832170883]

P_{12} = [5184894285212178249566461261834, 7390536788003150201273204464695859875505480483]

P_{13} = [-44691710236871465020671796121 66, 93106588928414589341332211373920814034144556 83]

P_{14} = [3606405835110925482450522970234, 2183644666981703632482662193390480040127898883]

P_{15} = [16151744576785317732688993162234, 61908882092472338946519909276455831463747210883]

P_{16} = [3573684355943766387962362869754, 2094467155115749424853047283659077805560259203]

P_{17} = [-7593760499388581664364916441 66, 8679171135458197195914024161800061810952119683]

P_{18} = [-53280587199358861821060031193 66, 69205881473794976332029355573674996762243500 83]

P_{19} = [5380268474895377355583039694554, 8105660240030025092450118297303424395856037443]

P_{20} = [17069233487425098088940203248484, 67583677272795299213867443505411893525786510633]

P_{21} = [5215432542403430758248050783794, 7501515746204716855921710958364078294243814643]

P_{22} = [2838942178046024039763692432122, 1212346280964590308944175800544505700108208003]

P_{23} = [243146882395382015946366404808154/81, 81162527216072633219928813618742750536658210810 7/729]

P_{24} = [2558229016839511149831260080762, 1706598395830079994387505244133382709649637123]

P_{25} = [2361253942905600810977556672634, 2157503396243552448798851089310708763298766083]

P_{26} = [2678312077644931683114439906234, 1462722361020796436741527433473386115047618883]

P_{27} = [3379397084927230910084852603902, 1608494167359575995485655188349208450365853755]

P_{28} = [3632407730870998917912491355514, 2255654937037700801978158381185619053396712963]

P_{29} = [2428778263277521959543043930234, 1998325023610603606161737305486867803334410883]

Previous record with rank ≥ 28

Fix a height $ht(E)$ e.g. naive height for a minimal Weierstrass model $E : y^2 = x^3 + Ax + B$ with no $p^4|A$ and $p^6|B$:

$$ht(E) := \max\{\, 4|A|^3,\ 27|B|^2 \,\}$$

$$\mathcal{E}(\leq X) := \{\, E/\mathbb{Q} \text{ up to } \mathbb{Q}\text{-isomorphism} : ht(E) \leq X \,\}$$

Conjecture (Goldfeld; Katz–Sarnak (rank distribution))

As $X \to \infty$,

$$\frac{|\{E/\mathbb{Q} \in \mathcal{E}(\leq X) : r = 0\}|}{|\mathcal{E}(\leq X)|} \to \tfrac{1}{2}, \quad \frac{|\{E/\mathbb{Q} \in \mathcal{E}(\leq X) : r = 1\}|}{|\mathcal{E}(\leq X)|} \to \tfrac{1}{2},$$

and

$$\frac{|\{E/\mathbb{Q} \in \mathcal{E}(\leq X) : r \geq 2\}|}{|\mathcal{E}(\leq X)|} \to 0.$$

To talk about average, we first need $\mathcal{N}(\mathbb{Q},\ X) := |\mathcal{E}(\leq X)|$

The Shafarevich conjecture for algebraic curves

Q: How many algebraic curves over a global field are there?

A: Infinitely many, so fix invariants and bound the "bad places".

Q: Let K be a number field with ring of integers $\mathcal{O}_K$, and let S be a finite set of primes of $\mathcal{O}_K$. How many K-isomorphism classes of smooth projective curves X/K of genus g have *good reduction* at all primes $\mathfrak{p} \notin S$?

A: I. R. Shafarevich (ICM 1962): this set should be finite.

34 I. R. ŠAFAREVIČ

There exists only a finite number of fields of algebraic functions K/k of a given genus $g \neq 1$, the critical prime divisors of which belong to a given finite set S.

This result also holds, with suitable modifications, for fields of genus $g = 1$. For this we must, in addition to the genus, consider another invariant

Finiteness / Effectivity / Exact Counts

- **Shafarevich conjecture:** for each triple (K, g, S), there are only finitely many isomorphism classes of smooth projective curves C/K of genus g with good reduction outside S.
- **Parshin:** Shafarevich finiteness $\Rightarrow$ Mordell finiteness (via Parshin's covering construction). **Faltings (1983):** Shafarevich finiteness for abelian varieties over number fields $\Rightarrow$ Mordell's conjecture for curves over number fields.
- However, these results are *ineffective* they prove finiteness but they do not give an explicit list, or even a good upper bound.

Question

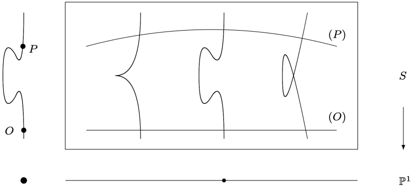
Can we go beyond finiteness and obtain exact counting formulas?

Answer

Yes, for $g = 1$ over $K = \mathbb{F}_q(t)$ via height moduli and its inertia.

Elliptic surfaces $/k$ = Families of elliptic curves $/K$

The study of **fibrations of curves** lies at the heart of the Enriques-Kodaira classification of algebraic surfaces.



Definition. An *elliptic surface* is a proper flat morphism

$$f : X \to C$$

from a smooth projective surface X to a smooth projective curve C such that the *geometric generic fiber* is a smooth curve of genus 1.

In this talk we focus on the standard arithmetic setup:

$$C = \mathbb{P}^1, \qquad \text{and } f \text{ admits a section } O : \mathbb{P}^1 \hookrightarrow X$$

(the *zero section*), which meets the smooth locus of f.

Geography $(K_X^2, \chi(\mathcal{O}_X)) = (0, n)$

Let $f : X \to \mathbb{P}^1$ be a *relatively minimal* elliptic surface with a section, and set $n := \chi(\mathcal{O}_X)$ (so $\deg \Delta = 12n$).

1. $n = 1$ **(rational elliptic surface).** Then X is rational and $\kappa(X) = -\infty$. Generically, Δ has 12 simple zeros, i.e. 12 nodal fibres (type I_1). One construction: take a pencil of plane cubics in $\mathbb{P}^2$ and blow up its 9 base points.

2. $n = 2$ **(elliptic $K3$ surface).** Then X is a projective $K3$ surface with an elliptic fibration, hence $\kappa(X) = 0$. Generically, Δ has 24 simple zeros, i.e. 24 nodal fibres; moduli dim $= 18$.

3. $n \geq 3$ **(minimal properly elliptic surface).** Then $\kappa(X) = 1$. Generically, Δ has $12n$ simple zeros, i.e. $12n$ nodal fibres.

Birkar's $\sigma(t)$ in the elliptic surface case ($A = S$)

Let $f\colon X \to \mathbb{P}^1$ be a height–n elliptic surface with section S. To compare with Birkar's setup for stable lc minimal models, take

$$B = 0, \qquad A = S, \qquad Z = \mathbb{P}^1, \qquad \kappa(K_X) = 1 \;\; (n \geq 3).$$

Birkar's associated polynomial is

$$\sigma(t) = (K_X + B + tA)^2 = (K_X + tS)^2 = K_X^2 + 2(K_X \cdot S)\, t + S^2 t^2.$$

For a relatively minimal elliptic surface of Kodaira dimension 1,

$$K_X^2 = 0, \qquad S^2 = -n, \qquad K_X \cdot S = n - 2,$$

hence

$$\boxed{\sigma(t) = 2(n-2)\, t - n\, t^2}$$

Deligne–Mumford stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

The stack $\overline{\mathcal{M}}_{1,1}$ is a smooth proper Deligne–Mumford stack parametrizing stable elliptic curves (genus 1 stable curves with a section). Its coarse moduli space, the j–line, is $\overline{M}_{1,1} \cong \mathbb{P}^1$.

Assume $\mathrm{char}(K) \neq 2, 3$. Then there is an isomorphism

$$(\overline{\mathcal{M}}_{1,1})_K \cong \left[\, (Spec K[a_4, a_6] \setminus \{(0,0)\})/\mathbb{G}_m \,\right] =: \mathcal{P}_K(4,6),$$

where $\mathbb{G}_m$ acts by $\lambda \cdot (a_4, a_6) = (\lambda^4 a_4, \lambda^6 a_6)$, corresponding to the short Weierstrass form

$$Y^2 = X^3 + a_4 X + a_6.$$

The special stabilizers occur at the orbifold points

$$[1:0] \text{ and } [0:1], \qquad \text{with stabilizers } \mu_4 \text{ and } \mu_6,$$

while a general point has stabilizer μ_2 (coming from the hyperelliptic involution on an elliptic curve).

Fix a field k. A **family of stable elliptic curves with a section** over $\mathbb{P}^1_k$ is equivalent to a morphism $\varphi : \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}$.

$$\begin{array}{ccccc} X & \xrightarrow{\nu} & Y := \varphi^*(\overline{\mathcal{E}}_{1,1}) & \longrightarrow & \overline{\mathcal{E}}_{1,1} \\ \downarrow f & & \downarrow g & & \downarrow p \\ \mathbb{P}^1 & = & \mathbb{P}^1 & \xrightarrow{\varphi} & \overline{\mathcal{M}}_{1,1} \end{array}$$

Here $p : \overline{\mathcal{E}}_{1,1} \to \overline{\mathcal{M}}_{1,1}$ is the universal stable elliptic curve, $g : Y \to \mathbb{P}^1$ is the pulled-back stable family, and $\nu : X \to Y$ is the relatively-minimal resolution (A_k singularity to I_{k+1} fiber).

Define the **height = degree** of φ to be $n := \deg\left(\varphi^*\mathcal{O}_{\overline{\mathcal{M}}_{1,1}}(1)\right)$ then the moduli stack of stable elliptic fibrations over the $\mathbb{P}^1$ with $Deg(\Delta) = 12n$ and a section is $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$.

Equivalently, writing $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$, a morphism φ is given by

$$\varphi = (a_4(u,v), a_6(u,v)), \qquad \deg(a_4) = 4n,\ \deg(a_6) = 6n,$$

with $\mathrm{Res}(a_4, a_6) \neq 0$ (so a_4, a_6 do not vanish simultaneously).

Projective elliptic K3 surface of height $n = 2$

$$y^2 = x^3 + a_4(u:v)x + a_6(u:v)$$

Weierstrass data for elliptic fibration on algebraic K3 surface,

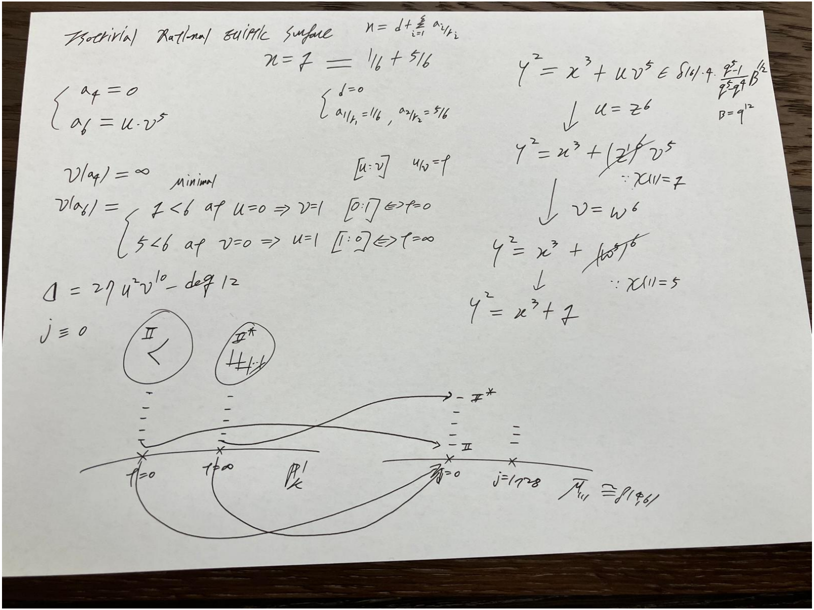
$$\begin{cases} a_4(u:v) & = -3u^4v^4, \\ a_6(u:v) & = u^5v^5(u^2+v^2). \end{cases}$$

Then we have $\Delta = 4a_4^3 + 27a_6^2$ and $j = 1728 \cdot 4a_4^3/\Delta$

$$\begin{cases} \Delta & = 27u^{10}v^{10}(u-v)^2(u+v)^2, \\ j & = 1728 \cdot -\frac{4u^2v^2}{(u-v)^2(u+v)^2}. \end{cases}$$

The j-map $j : \mathbb{P}^1 \to \overline{M}_{1,1} \cong \mathbb{P}^1$ is always a morphism but **lost the valuation data crucial for Tate's algorithm** to find out what are (additive) singular fibers at $[0:1]$ for $t = 0$ and $[1:0]$ for $t = \infty$.

Isotrivial rational elliptic surface of height $n = 1$

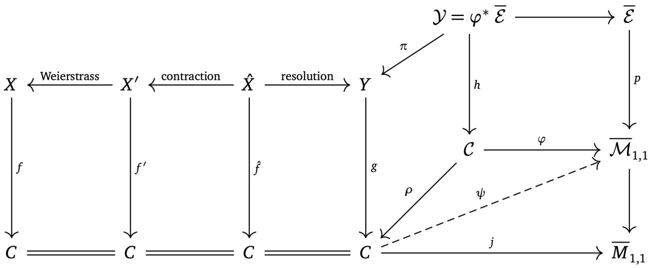


Vanishing $\gamma \Leftrightarrow$ Reduction type $\Theta \Leftrightarrow$ Twisting Γ

Theorem (Bejleri–Park–Satriano; April 2024)

If $\mathrm{char}(K) \neq 2, 3$. *Then the twisting condition* (r, a) *and the order of vanishing of* j *at* $j = \infty$ *determine the Kodaira fiber type, and* (r, a) *is in turn determined by* $m = \min\{3\nu(a_4), 2\nu(a_6)\}$.

$\gamma : (\nu(a_4),\ \nu(a_6))$	*Reduction type with* $j \in \overline{M}_{1,1}$	$\Gamma : (r, a)$
$(\geq 1, 1)$	II *with* $j = 0$	$(6, 1)$
$(1, \geq 2)$	III *with* $j = 1728$	$(4, 1)$
$(\geq 2, 2)$	IV *with* $j = 0$	$(3, 1)$
$(2, 3)$	$\mathrm{I}^*_{k>0}$ *with* $j = \infty$ I^*_0 *with* $j \neq 0, 1728$	$(2, 1)$
$(\geq 3, 3)$	I^*_0 *with* $j = 0$	$(2, 1)$
$(2, \geq 4)$	I^*_0 *with* $j = 1728$	$(2, 1)$
$(\geq 3, 4)$	IV^* *with* $j = 0$	$(3, 2)$
$(3, \geq 5)$	III^* *with* $j = 1728$	$(4, 3)$
$(\geq 4, 5)$	II^* *with* $j = 0$	$(6, 5)$



- $\psi\colon C \dashrightarrow \overline{\mathcal{M}}_{1,1}$: the *minimal* weighted linear series (rational j–map) $(a_4, a_6) \in H^0\big(C, \mathcal{O}(4n) \oplus \mathcal{O}(6n)\big)$ with $\gamma_x = (\nu_x(a_4), \nu_x(a_6))$, *minimal in the sense that there is no point $x \in C$ with $\nu_x(a_4) \geq 4$ and $\nu_x(a_6) \geq 6$.*
- $\varphi\colon \mathcal{C} \to \overline{\mathcal{M}}_{1,1}$ is the associated *representable twisted morphism* with twisting data $\Gamma = (r, a)$, obtained by taking root stacks at the indeterminacy points of ψ (via Bejleri–Park–Satriano correspondence). It induces a unique stable family over the stacky curve $h\colon \mathcal{Y} \to \mathcal{C}$ and, by passing to coarse moduli spaces, a twisted surface model $g\colon Y \to C$.
- $\hat{f}\colon \hat{X} \to C$ is a resolution of singularities of Y. Contract the relative (-1)–curves in $\hat{f}\colon \hat{X} \to C$ to obtain the relatively-minimal elliptic surface $f'\colon X' \to C$ (with the prescribed Kodaira fibres). Its relative log-canonical model is the Weierstrass model $f\colon X \to C$.

Proper polarized cyclotomic stacks

- A separated algebraic stack $\mathcal{X}$ of finite type over k is *cyclotomic* if every geometric stabilizer is cyclic:

$$Aut(\bar{x}) \;\simeq\; \mu_r \qquad (r \geq 1)$$

- $\mathcal{L}$ is *uniformizing* if each stabilizer acts faithfully on $\mathcal{L}|_{\bar{x}}$, i.e.

$$Aut(\bar{x}) \hookrightarrow \mathbb{G}_m$$

  Equivalently, the classifying map $\mathcal{X} \to B\mathbb{G}_m$ is representable.
- Let $\pi : \mathcal{X} \to X$ be the coarse moduli space. For a uniformizing $\mathcal{L}$, there exists $M \geq 1$ and a line bundle L on X such that

$$\mathcal{L}^{\otimes M} \;\cong\; \pi^* L$$

- $\mathcal{L}$ is *polarizing* if one can choose M so that L is *ample* on X. Then $(\mathcal{X}, \mathcal{L})$ is a *polarized cyclotomic stack*.
- *Example:* For a locally closed substack $\mathcal{X} \subset \mathcal{P}(\mathbf{w})$, the pullback $\mathcal{O}_{\mathcal{P}(\mathbf{w})}(1)|_{\mathcal{X}}$ is polarizing.

- $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ **over** $\mathbb{Z}\left[\frac{1}{6}\right]$ then Hodge bundle is $\lambda \simeq \mathcal{O}_{\mathcal{P}(4,6)}(1)$

$$\lambda^{\otimes 12} \;\cong\; c^*\mathcal{O}_{\mathbb{P}^1}(1), \qquad \bigl(12 = \mathrm{lcm}(4,6)\bigr)$$

- **Elliptic curves with level structure** Genus 0 modular curves:

  $\overline{\mathcal{M}}_{1,1}[\Gamma_1(2)] \cong \mathcal{P}(2,4)$, $\overline{\mathcal{M}}_{1,1}[\Gamma_1(3)] \cong \mathcal{P}(1,3)$, $\overline{\mathcal{M}}_{1,1}[\Gamma_1(4)] \cong \mathcal{P}(1,2)$,

  and for $m \in \{5,6,7,8,9,10,12\}$ one has $\overline{\mathcal{M}}_{1,1}[\Gamma_1(m)] \cong \mathbb{P}^1$ (over $Spec(\mathbb{Z}[1/m])$); also $\overline{\mathcal{M}}_{1,1}[\Gamma(2)] \cong \mathcal{P}(2,2)$

- **Genus** 1 **with marked points (Smyth stability).** Examples include

$$\mathcal{M}_{1,2}(1) \cong \mathcal{P}(2,3,4), \quad \mathcal{M}_{1,3}(2) \cong \mathcal{P}(1,2,2,3),$$

$$\mathcal{M}_{1,4}(3) \cong \mathcal{P}(1,1,1,2,2), \quad \mathcal{M}_{1,5}(4) \cong \mathbb{P}^5$$

- **Hyperelliptic genus** $g \geq 2$**.** The stack of monic odd-degree hyperelliptic curves is

$$\mathcal{H}_{2g}[2g{-}1] \;\cong\; \mathcal{P}(4,6,8,\ldots,4g{+}2)$$

  (in $\mathrm{char}(K) = 0$, and in $\mathrm{char}(K) > 2g{+}1$)

Height moduli stacks on cyclotomic stacks

Fix a smooth projective curve C/k with function field $K = k(C)$, a proper polarized cyclotomic stack $(\mathcal{X}, \mathcal{L})$ over a perfect field k.

Theorem (Bejleri–Park–Satriano; April 2024)

For each $n \in \mathbb{Z}_{\geq 0}$ there is a separated Deligne–Mumford stack $\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$ of finite type over k (with quasi-projective coarse moduli space) equipped with a canonical bijection

$$\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})(k) \; = \; \{ \, P \in \mathcal{X}(K) \; : \; \mathrm{ht}_{\mathcal{L}}(P) = n \, \}.$$

In particular, finite type is a geometric incarnation of the Northcott property for the stacky height.

1. There is a finite locally closed stratification

$$\bigsqcup_{\Gamma,d} \mathcal{H}^{\Gamma}_{d,C}(\mathcal{X},\mathcal{L})/S_{\Gamma} \to \mathcal{M}_{n,C}(\mathcal{X},\mathcal{L})$$

   where $\mathcal{H}^{\Gamma}_{d,C}$ are moduli spaces of twisted maps and the union runs over all possible admissible local conditions

$$\Gamma = (\{r_1, a_1\}, \ldots, \{r_s, a_s\})$$

   and degrees d for a twisted map to $(\mathcal{X},\mathcal{L})$ satisfying

$$n = d + \sum_{i=1}^{s} \frac{a_i}{r_i}$$

   and S_{Γ} is a subgroup of the symmetric group on s letters that permutes the stacky points of the twisted map.
2. Under the bijection in part (1), each k-point of $\mathcal{H}^{\Gamma}_{d,C}(\mathcal{X},\mathcal{L})/S_{\Gamma}$ corresponds to a K-point P with the stable height and local contributions given by

$$\mathrm{ht}^{st}_{\mathcal{L}}(P) = d \qquad \left\{\delta_i = \frac{a_i}{r_i}\right\}_{i=1}^{s}.$$

Theorem (Bejleri–Park–Satriano; April 2024)

Let $f\colon C \dashrightarrow \mathcal{P}(\vec{\lambda})$ be a rational map of smooth projective curve C, and let $P \in \mathcal{P}_C(\vec{\lambda})(K)$ denote the corresponding rational point over $K = k(C)$. Let $\{x_j\}$ be the indeterminacy points of f.

1. *Let $(L, s_0, \ldots, s_N)$ be any $\vec{\lambda}$-weighted linear series inducing f. Then the universal tuning stack $(\mathcal{C}, \pi, \overline{P})$ of P is the root stack of C obtained by taking the r_j-th root at x_j, where $r_j = r_{\min}(x_j; L, s_0, \ldots, s_N)$. Moreover, the induced morphism on stabilizers over x_j is given by the character $\chi_j^{-a_j}$ where $a_j = a_{\min}(x_j, L, s_0, \ldots, s_N)$.*
2. *A wls $(L, s_0, \ldots, s_N)$ is minimal if for each indeterminacy point $x \in C$, there exists an j such that $\nu_x(s_j) < \lambda_i$. There exists a unique minimal $\vec{\lambda}$-weighted linear series inducing f.*
3. *The stacky height $\mathrm{ht}_{\mathcal{O}(1)}(P)$ is equal to $\deg L$ where $(L, s_0, \ldots, s_N)$ is the unique minimal linear series. Moreover, the stable height is given by $\mathrm{ht}^{st}_{\mathcal{O}(1)}(P) = \deg \overline{P}^{*}\mathcal{O}(1)$ and the local contribution at x_j is given by $\delta_{x_j}(P) = \frac{a_j}{r_j}[k(x_j) : k]$.*

Specializing to the canonical case $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4,6)$

1. **Rational points = Minimal weighted linear series.** For $K = k(C)$, an elliptic curve E/K is a K–point of $\overline{\mathcal{M}}_{1,1}$, i.e. a minimal Weierstrass data $(a_4, a_6) \in H^0\big(C, \mathcal{O}(4n) \oplus \mathcal{O}(6n)\big)$ with no $x \in C$: $\nu_x(a_4) \geq 4$ and $\nu_x(a_6) \geq 6$.

2. **Minimal weighted linear series = Twisted morphisms.** Vanishing orders at a point $p \in C$,

$$\gamma_p = (\nu_p(a_4), \nu_p(a_6)),$$

   determine twisting data $\Gamma_p = (r_p, a_p)$ and hence the $\mathcal{C} \to C$.

3. **MMP dictionary (the bridge to Birkar).** Fixing vanishing profile γ (resp. twisting profile Γ) gives strata $\mathcal{W}^{\gamma}_{n,C}$ (resp. $\mathcal{H}^{\Gamma}_{d,C}$), which parameterize canonical/stable models of the associated elliptic surfaces with a specified singular fibers.

   Equivalently: this is the surface MMP for the pair (X, cS) (wall-crossing in c) packaged in moduli.

8 Different Types of Additive Bad Reductions

Let $\kappa := \operatorname{lcm}\{\lambda_0, \ldots, \lambda_N\}$ and $\bar{\lambda}_j := \kappa/\lambda_j$ as usual.

Lemma (Dori Bejleri–JP–Matthew Satriano; April 2024)

Suppose $\kappa > 1$. Then the map

$$m \mapsto \left(\frac{\kappa}{\gcd(m,\kappa)}, \frac{m}{\gcd(m,\kappa)}\right)$$

induces a bijection from the set $\{1, \ldots, \kappa - 1\}$ to the set

$$\{(r,a) : 1 \leq a < r,\ r|\kappa,\ \gcd(r,a) = 1\}$$

For $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ we have $\kappa = 12$ which means we have $m \in \{2,3,4,6,8,9,10\}$ ($\{1,5,7,11\}$ are excluded as prime) that corresponds to following rooting data $m = 2 \mapsto \frac{1}{6}$, $m = 3 \mapsto \frac{1}{4}$, $m = 4 \mapsto \frac{1}{3}$, $m = 6 \mapsto \frac{1}{2}$, $m = 8 \mapsto \frac{2}{3}$, $m = 9 \mapsto \frac{3}{4}$, $m = 10 \mapsto \frac{5}{6}$ which correspond to $7 + 1$ types of additive reductions.
($+1$ since ramification at $j = \infty$ for $\mathrm{I}^*_{k>0}$)

Suppose that normalized base multiplicity $m = 3$. This occurs if and only if $(\nu(a_4), \nu(a_6)) = (1, \geq 2)$. Then $r = 12/\gcd(3,12) = 4$ and $a = 3/\gcd(3,12) = 1$. Thus the stabilizer of the twisted curve acts on the central fiber of the twisted model via the character $\mu_4 \to \mu_4$, $\zeta_4 \mapsto \zeta_4^{-1}$. In particular, the central fiber E of $\mathcal{Y}$ has $j = 1728$. The μ_4 action on E has two fixed points, and there is an orbit of size two with stabilizer $\mu_2 \subset \mu_4$. Let E_0 be the image of E in the twisted model Y. As E appears with multiplicity 4, Y has $\frac{1}{4}(-1,-1)$ quotient singularities at the images of the the fixed points and a $\frac{1}{2}(-1,-1)$ singularity at the image of the orbit of size two. Each of these singularities is resolved by a single blowup to obtain $\hat{X}$ with central fiber $4\tilde{E}_0 + E_1 + E_2 + E_3$ where E_i are the exceptional divisors of the resolution for $i = 1,2,3$ and $E_1^2 = E_2^2 = -4$ with $E_3^2 = -2$. Then $\tilde{E}_0$ is a (-1)-curve so it needs to be contracted. After this contraction E_2 becomes a (-1) curve and must also be contracted. Since E_i for $i = 1,2,3$ are incident and pairwise transverse after blowing down $\tilde{E}_0$, then the images of E_1 and E_2 must be tangent after blowing down E_3. Moreover, they are now (-2)-curves and the relatively minimal model for type III.

Geometric Meaning of Height Moduli Framework

1. So one can run the resolution / minimal model. As these are *algebraic surfaces* it can be done over $\mathrm{char}(K) = p > 0$

2. A twisted morphism $\varphi : \mathcal{C} \to \overline{\mathcal{M}}_{1,1}$ with its twisting data Γ from the universal tuning stack $\mathcal{C}$ induces a stable stack-like model $h : \mathcal{Y} \to \mathcal{C}$ as a unique pullback of the universal family $p : \overline{\mathcal{E}} \to \overline{\mathcal{M}}_{1,1}$. All the ensuing birational geometry is natural.

3. True purpose of a **representable classifying morphism** is in the universal principle that φ intrinsically contains all the algebro-geometric data necessary to uniquely determine a fibration with singular fibers. This is the very essence of the inner arithmetic of rational points on moduli stacks over K.

1. Consider the fact that $\overline{\mathcal{M}}_{1,1}$ could have been any other algebraic stack $\mathcal{X}$ (such as $\overline{\mathcal{M}}_g$ or $\overline{\mathcal{A}}_g$) which is the representing object for certain moduli functor as the fine moduli stack together with the universal family $p : \overline{\mathcal{E}} \to \mathcal{X}$.

2. Representable classifying morphisms as twisted morphisms $\varphi : \mathcal{C} \to \mathcal{X}$ uniquely determines certain families of varieties (of algebraic curves or abelian varieties) with non-abelian stabilizers ($g \geq 2$). And they naturally have corresponding "Tate's algorithm", counting statements and so on.

3. Geometrizing $\mathcal{X}(K)$ leads to Height moduli space $\mathcal{M}_n(\mathcal{X}, \mathcal{V})$. Once we have a **space**, we compute its **invariants**, consider all invariants simultaneously via generating series and show the motivic height zeta function's **rationality**, naturally having various kinds of **consequences**.

Arithmetic of algebraic stacks over finite fields

- The *weighted point count* of a finite-type alg. stack $\mathcal{X}/\mathbb{F}_q$ is

$$\#_q(\mathcal{X}) := \sum_{[x] \in \mathcal{X}(\mathbb{F}_q)/\cong} \frac{1}{|Aut(x)|}$$

- We also want the actual count of isomorphism classes

$$|\mathcal{X}(\mathbb{F}_q)/\cong|$$

  which is immune to the Grothendieck-Lefschetz trace formula.

The *inertia stack* $\mathcal{I}\mathcal{X}$ parametrizes pairs $(x, Aut(x))$

Theorem (Changho Han–Park)

Let $\mathcal{X}/\mathbb{F}_q$ be an algebraic stack of finite type with quasi-separated finite type diagonal. Then

$$\#_q(\mathcal{I}\mathcal{X}) = |\mathcal{X}(\mathbb{F}_q)/\cong|.$$

Ekedahl in 2009 introduced the Grothendieck ring $K_0(\mathrm{Stck}_k)$

Definition (Ekedahl)

Fix a field k. The Grothendieck ring $K_0(\mathrm{Stck}_k)$ is the abelian group generated by isomorphism classes $\{\mathcal{X}\}$ of finite-type algebraic stacks over k with affine stabilizers, modulo:

- **Scissor relation:** if $\mathcal{Z} \subset \mathcal{X}$ is closed, then
$$\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\};$$
- **Vector bundle relation:** if $\mathcal{E} \to \mathcal{X}$ is a rank n vector bundle, then
$$\{\mathcal{E}\} = \{\mathcal{X} \times \mathbb{A}^n\}.$$

Multiplication is induced by products: $\{\mathcal{X}\} \cdot \{\mathcal{Y}\} := \{\mathcal{X} \times \mathcal{Y}\}$.

Let $\mathbb{L} := \{\mathbb{A}^1\}$ (the *Lefschetz motive*). Then
$$\{\mathbb{P}^N\} = 1 + \mathbb{L} + \cdots + \mathbb{L}^N, \qquad \{\mathbb{G}_m\} = \mathbb{L} - 1.$$

Universal for additive & multiplicative invariants

For any ring R and any function $\tilde{\nu} : \mathrm{Stck}_k \to R$ satisfying relations
1) $\tilde{\nu}(\mathcal{X}) = \tilde{\nu}(\mathcal{Y})$ whenever $\mathcal{X} \cong \mathcal{Y}$,
2) $\tilde{\nu}(\mathcal{X}) = \tilde{\nu}(\mathcal{U}) + \tilde{\nu}(\mathcal{X} \setminus \mathcal{U})$ for $\mathcal{U} \hookrightarrow \mathcal{X}$ an open immersion,
2) $\tilde{\nu}(\mathcal{X} \times \mathcal{Y}) = \tilde{\nu}(\mathcal{X}) \cdot \tilde{\nu}(\mathcal{Y})$,
there is a unique ring homomorphism $\nu : K_0(\mathrm{Stck}_k) \to R$

$$\begin{array}{ccc} & \mathrm{Stck}_k & \\ {\scriptstyle \{\ \}} \swarrow & & \searrow {\scriptstyle \tilde{\nu}} \\ K_0(\mathrm{Stck}_k) & \xrightarrow[\nu]{} & R \end{array}$$

Such homomorphism ν are called **motivic measures**.

$\therefore$ When $k = \mathbb{F}_q$, the point counting measure $\{\mathcal{X}\} \mapsto \#_q(\mathcal{X})$ is a well-defined ring homomorphism $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \to \mathbb{Q}$ giving the weighted point count $\#_q(\mathcal{X})$ of $\mathcal{X}$ over $\mathbb{F}_q$.

$\therefore$ When $k = \mathbb{C}$, χ_c and the Hodge–Deligne polynomial $E(-; u, v)$.

How many elliptic curves over $k = \mathbb{F}_q$ upto isom?

The inertia stack $\mathcal{I}\mathcal{M}_{1,1}$ parametrizes $[E]$ and automorphism groups $([E], \mathrm{Aut}[E])$. To keep track of the primitive roots of unity contained in $\mathbb{F}_q$, define function $\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$

$$\{\mathcal{M}_{1,1}\} = \mathbb{L}$$

For the inertia stack $\mathcal{I}\mathcal{M}_{1,1}$ for $\mathrm{char}(k) \neq 2, 3$ is equal to

$$\{\mathcal{I}\mathcal{M}_{1,1}\} = 2\mathbb{L} + \delta(6) \cdot 4 + \delta(4) \cdot 2$$

which translates to the following for $k = \mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$

$$\begin{aligned} \{\mathcal{I}\mathcal{M}_{1,1}\} &= 2\mathbb{L} + 6, \quad \text{if } q \equiv 1 \mod 12, \\ &= 2\mathbb{L} + 2, \quad \text{if } q \equiv 5 \mod 12, \\ &= 2\mathbb{L} + 4, \quad \text{if } q \equiv 7 \mod 12, \\ &= 2\mathbb{L}, \quad \text{if } q \equiv 11 \mod 12. \end{aligned}$$

Sharp enumerations over rational function field

Define *height of discriminant* Δ *over* $\mathbb{F}_q(t)$ as $ht(\Delta) := q^{\deg \Delta}$

Elliptic case: $Deg(\Delta) = 12n \implies ht(\Delta) = q^{12n}$ for $n \in \mathbb{Z}_{\geq 0}$

$$\mathcal{N}(\mathbb{F}_q(t), B) := \left| \left\{ E/\mathbb{F}_q(t) \text{ up to } \mathbb{F}_q(t)\text{-isomorphism} : 0 < ht(\Delta) \leq B \right\} \right|$$

Theorem (Bejleri–Park–Satriano; April 2024)

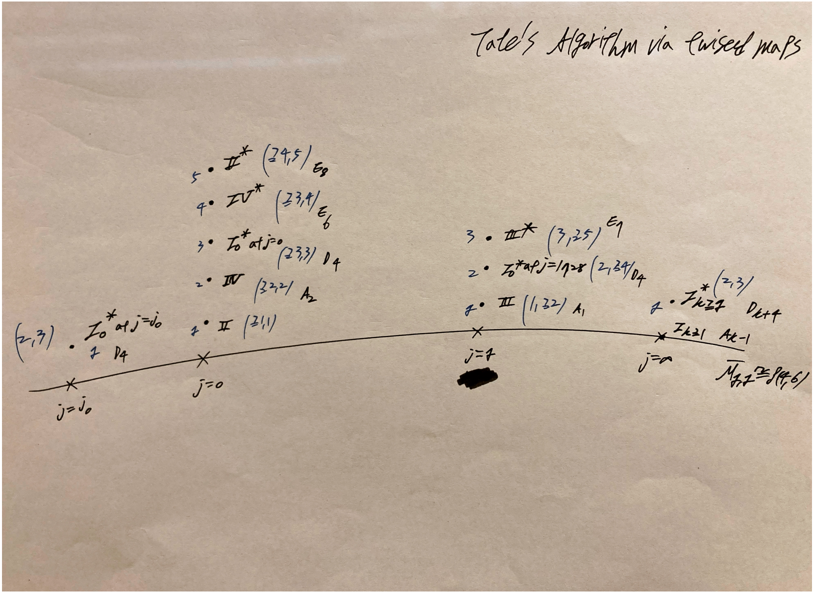
Let $\mathrm{char}(\mathbb{F}_q) > 3$ and $\delta(x) := \begin{cases} 1 & \text{if } x \text{ divides } q-1, \\ 0 & \text{otherwise.} \end{cases}$, then

$$\begin{aligned} \mathcal{N}(\mathbb{F}_q(t),\ B) = {} & 2\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6} - 2B^{1/6} \\ & + \delta(6)\cdot 4\left(\frac{q^5-1}{q^5-q^4}\right)B^{1/2} + \delta(4)\cdot 2\left(\frac{q^3-1}{q^3-q^2}\right)B^{1/3} \\ & + \delta(6)\cdot 4 + \delta(4)\cdot 2 \end{aligned}$$

Origins of the terms in $\mathcal{N}(\mathbb{F}_q(t), B)$

- $2\left(\frac{q^9-1}{q^8-q^7}\right) B^{5/6}$: non-constant μ_2–twist families that are either *non-isotrivial*, or *isotrivial with* $j \neq \infty$.
- $-2\,B^{1/6}$: non-constant μ_2–twist families of *generically singular isotrivial* elliptic curves with $j = \infty$.
- $\delta(6) \cdot 4\left(\frac{q^5-1}{q^5-q^4}\right) B^{1/2}$: non-constant μ_6–twist families of *isotrivial* elliptic curves with $j = 0$.
- $\delta(4) \cdot 2\left(\frac{q^3-1}{q^3-q^2}\right) B^{1/3}$: non-constant μ_4–twist families of *isotrivial* elliptic curves with $j = 1728$.
- $\delta(6) \cdot 4$: *constant* elliptic curves with $j = 0$.
- $\delta(4) \cdot 2$: *constant* elliptic curves with $j = 1728$.

Geometric Tate's algorithm



Sieving by minimality on ambient projective stacks

- A weighted linear series may fail to be *minimal* (so it does *not* represent a height-n rational point). The *minimality defect* e measures how far it is from being minimal.

- **Quotient–remainder of the base profile.** Let $\mu = (\mu_i)$ be the normalized base profile and fix κ (the minimality threshold). Write uniquely

$$\mu_i = \kappa q_i + r_i, \qquad q_i \in \mathbb{Z}_{\geq 0}, \quad 0 \leq r_i < \kappa.$$

  Define

$$q(\mu) := (q_i), \qquad r(\mu) := (r_i), \qquad e := |q(\mu)| = \sum_i q_i.$$

- **Sieve viewpoint.** We start from the ambient parameter space of *all* weighted linear series (including non-minimal ones), and *sieve out* the bad locus by stratifying according to the defect e:

$$\text{ambient} \;=\; \bigsqcup_{e \geq 0} (\text{defect } e), \qquad \text{minimal locus} \;=\; (\text{defect } 0).$$

  Motivic analogue of inclusion–exclusion: we control the complement of the minimal locus by understanding the defect strata.

Algebraization via motives & Run Euler product

Corollary (Bejleri–Park–Satriano; April 2024)

After finite constructible stratification of source and target, the map

$$\psi_n : \bigsqcup_{e=0}^{n} \mathcal{W}_{n-e}^{min} \times \mathbb{P}(V_e^1) \to \mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_j}\right)$$

is an isomorphism on each stratum (hence induces equality of motives).

- **Moral.** For *additive* invariants, we can replace a space by any other space with the *same* motivic class:

  $\{X\} = \{Y\}$ in $K_0(\mathrm{Stck}_k) \quad \Longrightarrow \quad \nu(X) = \nu(Y)$ for every motivic measure

- **How this is used.** To compute an arithmetic invariant of X, stratify into locally closed pieces $X = \bigsqcup_i X_i$ and add:

  $$\{X\} = \sum_i \{X_i\}.$$

  If X is hard, replace it by a stratified-isomorphic Y whose pieces are computable, and use $\{X\} = \{Y\}$.

Motivic Height Zeta Function as Generating Series

Definition

The motivic height zeta function of $\mathcal{P}(\lambda_0, \ldots, \lambda_N)$ is the formal power series

$$Z_{\vec{\lambda}}(t) := \sum_{n \geq 0} \{\mathcal{W}_n^{min}\} \, t^n \in K_0(\mathrm{Stck})[\![t]\!]$$

where $\mathcal{W}_n^{min}$ is the space of minimal weighted linear series on $\mathbb{P}^1$ of height n. We also define the variant

$$\mathcal{I}Z_{\vec{\lambda}}(t) := \sum_{n \geq 0} \{\mathcal{I}\mathcal{W}_n^{min}\} \, t^n \in K_0(\mathrm{Stck}_k)[\![t]\!]$$

1. We denote the usual motivic zeta function of $\mathbb{P}^1$ by

$$Z(t) = \sum \{\mathrm{Sym}^e \mathbb{P}^1\} t^e = \frac{1}{(1 - \mathbb{L}t)(1 - t)}$$

2. **We stratify by minimality defect** e to obtain an equality

$$\left\{\mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_i}\right)\right\} = \sum_{e=0}^{n} \{\mathcal{W}_{n-e}^{min}\}\{\mathrm{Sym}^e \mathbb{P}^1\}$$

which implies

$$\sum_{n \geq 0} \left\{\mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_i}\right)\right\} t^n = Z_{\vec{\lambda}}(t) \cdot Z(t) \qquad (1)$$

3. *Homogeneous polynomials* live in compact ambient stack!

$$\sum_{n \geq 0} \left\{\mathcal{P}\left(\bigoplus_{i=0}^{N} V_n^{\lambda_i}\right)\right\} t^n = \frac{\{\mathbb{P}^N\} + \mathbb{L}^{N+1}\{\mathbb{P}^{|\vec{\lambda}|-N-2}\} t}{(1 - t)(1 - \mathbb{L}^{|\vec{\lambda}|} t)}$$

Rationality of Motivic Height Zeta Function

Fix weights $\vec{\lambda} = (\lambda_0, \ldots, \lambda_N)$ and let $|\vec{\lambda}| := \sum_{i=0}^{N} \lambda_i$. Suppose for simplicity that k contains all $\mathrm{lcm} = \mathrm{lcm}(\lambda_0, \ldots, \lambda_N)$ roots of unity.

Theorem (Bejleri–Park–Satriano; April 2024)

For $k, \vec{\lambda}$ as above and $C = \mathbb{P}^1_k$, consider $\mathcal{W}_n^{min}$ and its inertia stack $\mathcal{I}\mathcal{W}_n^{min}$. We have the following formulas over $K_0(\mathrm{Stck}_k)$.

$$\sum_{n\geq 0} \{\mathcal{W}_n^{min}\} t^n = \frac{1 - \mathbb{L}t}{1 - \mathbb{L}^{|\vec{\lambda}|} t} \left(\{\mathbb{P}^N\} + \mathbb{L}^{N+1} \{\mathbb{P}^{|\vec{\lambda}| - N - 2}\} t \right)$$

$$\sum_{n\geq 0} \{\mathcal{I}\mathcal{W}_n^{min}\} t^n = \sum_{g \in \mu_{\mathrm{lcm}}(k)} \frac{1 - \mathbb{L}t}{1 - \mathbb{L}^{|\vec{\lambda}_g|} t} \left(\{\mathbb{P}^{N_g}\} + \mathbb{L}^{N_g+1} \{\mathbb{P}^{|\vec{\lambda}_g| - N_g - 2}\} t \right)$$

where g runs over the lcm roots of unity and $\vec{\lambda}_g$ is a subset of $\vec{\lambda}$ of size $N_g + 1$ depending explicitly on the order of g.

Theorem (Bejleri–Park–Satriano; April 2024)

$$\left\{\mathcal{W}^{\min}_{n=1}(\mathcal{P}(\vec{\lambda}))\right\} = \{\mathbb{P}^N\}(\mathbb{L}^{|\vec{\lambda}|} - \mathbb{L}) + \mathbb{L}^{N+1}\{\mathbb{P}^{|\vec{\lambda}|-N-2}\}$$
$$\left\{\mathcal{W}^{\min}_{n\geq 2}(\mathcal{P}(\vec{\lambda}))\right\} = \mathbb{L}^{(n-2)|\vec{\lambda}|+N+2}(\mathbb{L}^{|\vec{\lambda}|-1} - 1)\{\mathbb{P}^{|\vec{\lambda}|-1}\}$$

Take $|\vec{\lambda}| = 10$ and $N = 1$ as $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $\mathbb{Z}[1/6]$.

1. When $n = 1$, X is a **Rational elliptic surface**.

$$\left\{\mathcal{W}^{\min}_1\right\} = \mathbb{L}^{11}+\mathbb{L}^{10}+\mathbb{L}^{9}+\mathbb{L}^{8}+\mathbb{L}^{7}+\mathbb{L}^{6}+\mathbb{L}^{5}+\mathbb{L}^{4}+\mathbb{L}^{3}-\mathbb{L}$$

2. When $n = 2$, X is algebraic $K3$ surface with elliptic fibration (i.e., **Projective elliptic K3 surface with moduli dim. 18**).

$$\left\{\mathcal{W}^{\min}_2\right\} = \mathbb{L}^{21}+\mathbb{L}^{20}+\mathbb{L}^{19}+\mathbb{L}^{18}+\mathbb{L}^{17}+\mathbb{L}^{16}+\mathbb{L}^{15}+\mathbb{L}^{14}+\mathbb{L}^{13}-\mathbb{L}^{11}-\mathbb{L}^{10}-\mathbb{L}^{9}-\mathbb{L}^{8}-\mathbb{L}^{7}-\mathbb{L}^{6}-\mathbb{L}^{5}-\mathbb{L}^{4}-\mathbb{L}^{3}$$

$$= \mathbb{L}(\mathbb{L}^2-1)\left(\mathbb{L}^{18}+\mathbb{L}^{17}+2\mathbb{L}^{16}+2\mathbb{L}^{15}+3\mathbb{L}^{14}+3\mathbb{L}^{13}+4\mathbb{L}^{12}+4\mathbb{L}^{11}+5\mathbb{L}^{10}+4\mathbb{L}^{9}+4\mathbb{L}^{8}+3\mathbb{L}^{7}+3\mathbb{L}^{6}+2\mathbb{L}^{5}+2\mathbb{L}^{4}+\mathbb{L}^{3}+\mathbb{L}^{2}\right)$$

Motives of moduli stacks of elliptic surfaces

Theorem (Bejleri–Park–Satriano)

Let $\mathrm{char}(k) \neq 2, 3$. The motives (modulo $\{\mathsf{PGL}_2\}$) of moduli stacks $\mathcal{W}^{\Theta}_{\min,n}$ of minimal Weierstrass fibrations with a single Kodaira fiber Θ and at worst multiplicative reduction elsewhere is

Reduction type Θ with $j \in \overline{M}_{1,1}$	$\lvert\gamma\rvert$	$\{\mathcal{W}^{\Theta}_{\min,n}\} \in K_0(\mathrm{Stck}_K)$
$\mathrm{I}_{k>0}$ *with* $j = \infty$	*0*	$\mathbb{L}^{10n-2}$
II *with* $j = 0$	*2*	$\mathbb{L}^{10n-3}$
III *with* $j = 1728$	*3*	$\mathbb{L}^{10n-4}$
IV *with* $j = 0$	*4*	$\mathbb{L}^{10n-5}$
$\mathrm{I}^*_{k>0}$ *with* $j = \infty$ I^*_0 *with* $j \neq 0, 1728$	*5*	$\mathbb{L}^{10n-6} - \mathbb{L}^{10n-7}$
I^*_0 *with* $j = 0, 1728$	*6*	$\mathbb{L}^{10n-7}$
IV^* *with* $j = 0$	*7*	$\mathbb{L}^{10n-8}$
III^* *with* $j = 1728$	*8*	$\mathbb{L}^{10n-9}$
II^* *with* $j = 0$	*9*	$\mathbb{L}^{10n-10}$

Motivic Analytic Number Theory Praxis

Moduli of minimal stable $E/\mathbb{F}_q(t)$ is $\mathcal{L}_{12n} = \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$

Theorem (Changho Han–Park)

Grothendieck class in $K_0(\mathrm{Stck}_k)$ with $\mathrm{char}(k) \neq 2, 3$,

$$\{\mathcal{L}_{12n}\} = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$$

Weighted point count over $\mathbb{F}_q$ with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,

$$\#_q(\mathcal{L}_{12n}) = q^{10n+1} - q^{10n-1}$$

Exact number of $\mathbb{F}_q$–isomorphism classes with $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$,

$$|\mathcal{L}_{12n}(\mathbb{F}_q)/\sim| = \#_q(\mathcal{I}\mathcal{L}_{12n}) = 2 \cdot (q^{10n+1} - q^{10n-1})$$

$$\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) = \sum_{n=1}^{\left\lfloor \frac{log_q \mathcal{B}}{12} \right\rfloor} |\mathcal{L}_{1,12n}(\mathbb{F}_q)/\sim| = 2 \cdot \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot \left(\mathcal{B}^{\frac{5}{6}} - 1\right)$$

Theorem (Park; July 2025)

For $q = 3^r$, there are $4/6$ twists at supersingular $j = 0$

- r is odd :

$$\mathcal{N}(\mathbb{F}_q(t),\ B) = 2\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6} - 2B^{1/6} + 2\left(\frac{q^7-1}{q^7-q^6}\right)B^{2/3} - 2\left(\frac{q^3-1}{q^4-q^3}\right)B^{1/3}$$

- r is even :

$$\mathcal{N}(\mathbb{F}_q(t),\ B) = 2\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6} - 2B^{1/6} + 4\left(\frac{q^7-1}{q^7-q^6}\right)B^{2/3} - 4\left(\frac{q^3-1}{q^4-q^3}\right)B^{1/3}$$

Theorem (Park; July 2025)

For $q = 2^r$, there are 3/7 twists at supersingular $j = 0$

- r is odd :

$$\mathcal{N}(\mathbb{F}_q(t),\ B) = 2\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6} - 2B^{1/6} + \left(\frac{q^8-1}{q^8-q^7}\right)B^{3/4} - \left(\frac{q^5-1}{q^6-q^5}\right)B^{1/2} - 2q + 4$$

- r is even :

$$\mathcal{N}(\mathbb{F}_q(t),\ B) = 2\left(\frac{q^9-1}{q^8-q^7}\right)B^{5/6} - 2B^{1/6} + 5\left(\frac{q^8-1}{q^8-q^7}\right)B^{3/4} - 5\left(\frac{q^5-1}{q^6-q^5}\right)B^{1/2} - 2q + 4$$

Happy New Year!

Thank you!

Accessing Cruder Level of Topology via Motives

A priori, point counts over $\mathbb{F}_q$ shouldn't know any topology.

In $\mathbb{A}^2_k$, cusp singular fiber II and affine line $\mathbb{A}^1$ have the same point counts (motives) i.e. $\{\mathrm{II} = V(y^2 = x^3)\} = \mathbb{L} = \{\mathbb{A}^1 = V(x)\}$ but they have very different *topology*.

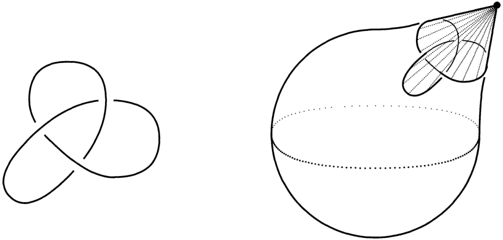
Same motive since we have a stratification of $\mathrm{II} = X_1 \cup X_2$ where $X_1 = \mathrm{II} - \{pt\}$ and $X_2 = \{pt\}$ and $\mathbb{A}^1 = Y_1 \cup Y_2$ where $Y_1 = \mathbb{A}^1 - \{pt\}$ and $Y_2 = \{pt\}$.

Indeed, $X_1 \cong Y_1$ (smooth complement) and $X_2 \cong Y_2$ (a singular point is just like a smooth point as $Spec(k)$) i.e. they are *cut-and-paste equivalent* and naturally $\{\mathrm{II}\} = \{\mathbb{A}^1\} = \mathbb{L}$

Same for nodal cubic $\{\mathrm{I}_1 = V(y^2 = x^3 + x^2)\} = \mathbb{L}$

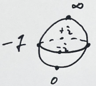
Different topology since, II and I_1 have arithmetic genus 1 (they are singular elliptic curves) whereas $\mathbb{A}^1$ has arithmetic genus 0

Singular point on II is the tip of a cone over the trefoil knot whereas singular point on I_1 is the tip of a cone over the Hopf link. (Every isolated singularity of a complex curve in a complex surface can be described topologically as the tip of a cone on a link)



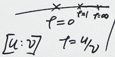
8.7. Trefoil knot, and cusp fiber

Miracle: When a variety is smooth projective then its point count over $\mathbb{F}_q$ knows topology via Frobenius weights and étale purity (the finite field analogue of RH) through the Grothendieck-Lefschetz trace formula under the Weil conjecture framework.



$r = \frac{1}{2}$, $Z = 4$
$Vol(\mathbb{C}P^1) = \pi$

$b_0 = 1$
$b_2 = 1$
$e = 2$



$|\mathbb{P}^1(\mathbb{F}_q)| = q+1$

$\mathbb{C}P^1$ — G.B.F — S^2 — G.L.T.F — $\mathbb{P}^1_{\mathbb{F}_q}$

Chow

W.C.

W.T.

$\mathbb{C}(z)$ - - - $\mathbb{F}_q(t)$ —∞— $\mathbb{F}_q(t)$ — Poly/Poly

Holo/Holo

G.F.A.

$\mathbb{Q}$ — Int/Int

AG/$\mathbb{C}$ AT/k AG/$\mathbb{F}_q$

KT

CV CA

NT

$\mathbb{Z} \sim \mathbb{F}_q[t]$ 'As integers so polynomials'

Let $\mathcal{E}$ be a suitable cat. of sheaves
then
$\mathcal{E}(\mathrm{Spec}\,\mathbb{Z} \setminus T) \sim \mathcal{E}(\mathbb{P}^1_{\mathbb{F}_q} \setminus S)$

Analogy

'Aware of each others'

V. Arnol'd, J. Milnor, M. Atiyah, G. Segal

1. Hom space $\mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$ is the moduli space of morphisms $f : \mathbb{P}^1_D \to \mathbb{P}^1_T$ of degree n as $f^*\mathcal{O}_{\mathbb{P}^1_T}(1) \cong L_{\mathbb{P}^1_D} \cong \mathcal{O}_{\mathbb{P}^1_D}(n)$.

2. A morphism $f : \mathbb{P}^1_D \to \mathbb{P}^1_T$ consists of global sections (global homogeneous polynomials) $f = (s_0(u : v), s_1(u : v))$ where $\deg(s_0) = \deg(s_1) = n$ and are coprime i.e. $\mathrm{Res}(s_0, s_1) \neq 0$.

3. Consider $f = (-27u^{12}v^{12}, 27u^{14}v^{10} - 54u^{12}v^{12} + 27u^{10}v^{14})$ is a **degree 4** morphism as the common factor is $27u^{10}v^{10}$

4. The rational maps and the morphisms coincide i.e. $f : \mathbb{P}^1_D \dashrightarrow \mathbb{P}^1_T = f : \mathbb{P}^1_D \to \mathbb{P}^1_T$ ($\mathbb{P}^1_D$ smooth $\mathbb{P}^1_T$ projective) **after cancellation of common factors** i.e. $\gcd(s_0, s_1) = 1$

5. $\mathbb{P}^1_T(k(t))_n = \mathbb{P}^1_T(k[t])_n$ for $\mathbb{P}^1_D$ with function field $k(t)$ and ring of integers $\mathcal{O}_{k(t)} = k[t] \sim \mathbb{P}^1_T(\mathbb{Q})_{ht(a/b)} = \mathbb{P}^1_T(\mathbb{Z})_{ht(a/b)}$

Arithmetic of $X_n := \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$

1. $X_n = \mathbb{P}^{2n+1} - V(\mathrm{Res}(s_0, s_1))$ is the open complement of **Resultant hypersurface** $\mathrm{Res}(s_0, s_1) = 0$ in $\mathbb{P}^{2n+1}$ thus it is an open quasiprojective variety of dimension $2n+1$

2. By Farb-Wolfson's seminal work (2016) $\{X_n\} = \mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} \to |X_n(\mathbb{F}_q)| = q^{2n+1} - q^{2n-1}$

3. Both domain $\mathbb{P}^1_D$ and target $\mathbb{P}^1_T$ are **unparameterized** and the action of an element of PGL_2 on the homogeneous coordinates $[u : v]$ of $\mathbb{P}^1_D$ translates to an action on the global sections s_i of $\mathcal{O}_{\mathbb{P}^1_D}(n)$ for $i = 0, 1$ which are the homogeneous coordinates of $\mathbb{P}(V) = \mathcal{P}(\underbrace{1, \ldots, 1}_{n+1 \text{ times}}, \underbrace{1, \ldots, 1}_{n+1 \text{ times}}) = \mathbb{P}^{2n+1}$

4. $\mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} = \mathbb{L}(\mathbb{L}^2 - 1) \cdot \mathbb{L}^{2n-2}$ as $\{\mathrm{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$

Topology of $X_n := \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$

1. $\mathrm{Hom}^*_n(\mathbb{P}^1_D, \mathbb{P}^1_T) \hookrightarrow \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T) \to \mathbb{P}^1_T$ via the evaluation morphism $\mathrm{ev}_\infty : \mathrm{Hom}_n(\mathbb{P}^1_D, \mathbb{P}^1_T) \to \mathbb{P}^1_T$ with $f \mapsto f(\infty) \in \mathbb{P}^1_T$
2. Fiber $\mathrm{Hom}^*_n(\mathbb{P}^1_D, \mathbb{P}^1_T)$ is the based mapping space which is identical to the space of coprime polynomials $\mathrm{Poly}_1^{(n,n)}$

Definition

Fix a field K with algebraic closure $\overline{K}$. Fix $k, l \geq 0$. Define $\mathrm{Poly}_1^{(k,l)}$ to be the set of pairs (u, v) of monic polynomials in $K[z]$ so that:

2.1 $\deg u = k$ and $\deg v = l$.

2.2 u and v have no common root in $\overline{K}$.

3. ev_∞ is a Zariski-locally trivial fibration via the transitive action of $Aut(\mathbb{P}^1_T) = \mathrm{PGL}_2$
4. $\mathbb{L}^{2n+1} - \mathbb{L}^{2n-1} = (\mathbb{L} + 1) \cdot (\mathbb{L}^{2n} - \mathbb{L}^{2n-1})$ as $\{\mathrm{Hom}^*_n(\mathbb{P}^1_D, \mathbb{P}^1_T)\} = \{\mathrm{Poly}_1^{(n,n)}\} = \mathbb{L}^{2n} - \mathbb{L}^{2n-1}$

Summary of Faltings' Proof by H. Darmon

Faltings' proof of Mordell's conjecture is based on a sequence of maps (here X is a curve of genus g defined over K and having good reduction outside of the finite set S of primes of K):

$$\left\{\begin{array}{l} K\text{-rational} \\ \text{points on } X \end{array}\right\} \xrightarrow{R_1} \left\{\begin{array}{l} \text{Curves of genus } g' \text{ over } K' \\ \text{with good reduction outside } S' \end{array}\right\}$$

$$\xrightarrow{R_2} \left\{\begin{array}{l} \text{Isomorphism classes of semistable} \\ \text{abelian varieties of dimension } g' \\ \text{with good reduction outside } S' \end{array}\right\}$$

$$\xrightarrow{R_3} \left\{\begin{array}{l} \text{Isogeny classes of abelian varieties} \\ \text{of dimension } g' \\ \text{with good reduction outside } S' \end{array}\right\}$$

$$\xrightarrow{R_4} \left\{\begin{array}{l} \text{Rational semisimple } \ell\text{-adic representations} \\ \text{of dimension } 2g' \text{ unramified outside } S'_\ell \end{array}\right\}$$

Construct the height moduli spaces $\mathcal{W}_{n,K}^{min}$ of rational points of height n on moduli stacks of algebraic curves, abelian varieties, and G-characters over global fields, and show that they are *'of finite type'*, followed by an analogue of *'Tate's algorithm'*.

1. The map R_1 is given by Parshin's construction, and is finite-to-one, by the geometric theorem of De Franchis.

2. The map R_2 is defined by passing to the jacobian of a curve, and is finite-to-one by Torelli's theorem.

3. The map R_3 is the obvious one, and is finite-to-one, by Falting's fundamental Theorem 2.11 on finiteness of abelian varieties in a given isogeny class.

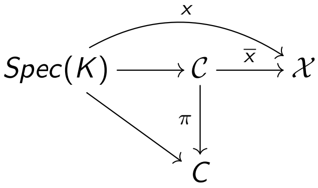
4. The map R_4 is defined by passing to the Tate module, and is one-to-one, thanks to the Tate conjectures proved by Faltings. The proof of the Tate conjectures is obtained by combining a strategy of Tate with the finiteness Theorem 2.11. These ideas are also used to show that the Galois representations arising in the image of R_4 are *semisimple*.

5. The last set in this sequence of maps is finite by the finiteness principle for rational semisimple ℓ-adic representations, which is itself a consequence of the Chebotarev density theorem and the Hermite–Minkowski theorem.

Better yet, show the *rationality of motivic height zeta functions* of height moduli spaces, followed by the extraction of coefficients.

Stacky Heights on Algebraic Stacks wrt 'Ample' $\mathcal{V}$

Ellenberg, Zureick-Brown, and Satriano extends the rational point $x \in \mathcal{X}(K)$ to a stacky curve, called a *tuning stack* $(\mathcal{C}, \pi, \overline{x})$ for x.



$\mathcal{C}$ is a normal, π is a birational coarse space map.

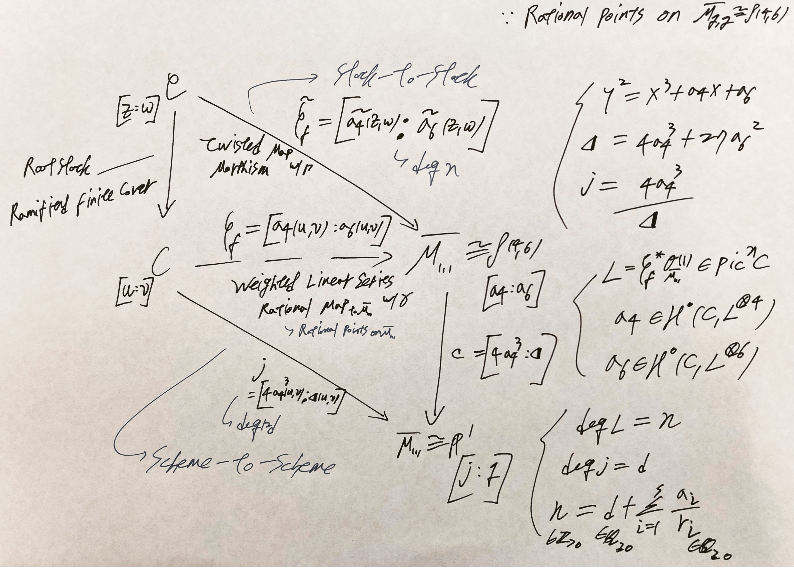
Definition

If $\mathcal{V}$ is a vector bundle on $\mathcal{X}$ and $x \in \mathcal{X}(K)$, the *height of x with respect to $\mathcal{V}$* is defined as

$$\mathrm{ht}_{\mathcal{V}}(x) := -\deg(\pi_* \overline{x}^* \mathcal{V}^\vee)$$

for any choice of tuning stack $(\mathcal{C}, \pi, \overline{x})$.

Rational points on $\overline{\mathcal{M}}_{1,1} \cong \mathcal{P}(4,6)$ over $K = k(C)$



Precise proportions of E/K motivated by NT

Theorem (Generic Torsion Freeness; Phillips)

The set of torsion-free elliptic curves over global function fields has density 1. i.e., 'Most elliptic curves over K are torsion free'.

Theorem (Boundedness; Tate-Shafarevich & Ulmer)

The ranks of <u>non-constant</u> elliptic curves over $\mathbb{F}_q(t)$ are unbounded (in both the **isotrivial** *and the* **non-isotrivial** *cases).*

Ulmer's non-isotrivial elliptic curve of infinite rank

1. Start with $y^2 + xy = x^3 - t^d$, then *complete the square* via $y = y' - \frac{x}{2}$ and then *complete the cubic* via $x = x' - \frac{1}{12}$. We need $\mathrm{char}(k) \neq 2, 3$ to get to the short Weierstrass form.

2. We get $y^2 = x^3 - \frac{1}{48}x + \frac{1}{864} - t^d$. Coefficients should be integral thus we take $\lambda = 2 \cdot 3$ to multiply λ^4 to $-\frac{1}{48}$ and λ^6 to $+\frac{1}{864} - t^d$.

3. We arrive at $y^2 = x^3 - 27x + 54 - 2^6 \cdot 3^6 \cdot t^d$ thus $[-\frac{1}{48} : \frac{1}{864} - t^d] = [-27 : 54 - 2^6 \cdot 3^6 \cdot t^d]$.

4. Remember the isomorphism, for any $\lambda \in \mathbb{G}_m$

$$\left[y^2 = x^3 + Ax + B\right] \cong \left[y^2 = x^3 + \lambda^4 \cdot Ax + \lambda^6 \cdot B\right]$$

via $x \mapsto \lambda^{-2} \cdot x$ and $y \mapsto \lambda^{-3} \cdot y$ by the *Weighted homogeneous coordinate* of $\mathcal{P}(4, 6)$.

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Enter your code in the box below. Click on "Submit" to have it evaluated by Magma.

```
KK<t> := FunctionField(GF(4007));
E := EllipticCurve([-27, 54 - 2^6*3^6*t^11]);
E;
&*BadPlaces(E);
LocalInformation(E);
```

Cancel Submit

```
Elliptic Curve defined by y^2 = x^3 + 3980*x + (1428*t^11 + 54) over Univariate
rational function field over GF(4007)
t^11 + 1549
[ <(t^5 + 3335*t^4 + 2186*t^3 + 488*t^2 + 2393*t + 906), 1, 1, 1, I1, false>,
<(t^5 + 3337*t^4 + 2186*t^3 + 488*t^2 + 3369*t + 906), 1, 1, 1, I1, false>,
<(t), 11, 1, 11, I11, true>, <(1/t), 2, 2, 1, II, true>, <(t + 1342), 1, 1, 1,
I1, false> ]
```

1. The corresponding elliptic surface has a fiber of Kodaira type I_d at zero (at $t = 0$), while the fiber at infinity (at $1/t = 0$) is given by the congruence class $\overline{d}$ of d modulo 6 : $(\overline{d}, \Theta)$ $(\overline{0}, \mathrm{I}_0)$ $(\overline{1}, \mathrm{II}^*)$ $(\overline{2}, \mathrm{IV}^*)$ $(\overline{3}, \mathrm{I}_0^*)$ $(\overline{4}, \mathrm{IV})$ $(\overline{5}, \mathrm{II})$

2. Outside $\mathrm{char}\ 2, 3$, there are d fibres of type I_1 at the zeroes of $432t^d - 1$ (some of which may be merged if $\mathrm{char}(k)|d$).

The aim of this paper is to produce elliptic curves over $K = \mathbb{F}_p(t)$ which are nonisotrivial ($j \notin \mathbb{F}_p$) and which have arbitrarily large rank.

THEOREM 1.5. *Let p be an arbitrary prime number, $\mathbb{F}_p$ the field of p elements, and $\mathbb{F}_p(t)$ the rational function field in one variable over $\mathbb{F}_p$. Let E be the elliptic curve defined over $K = \mathbb{F}_p(t)$ by the Weierstrass equation*

$$y^2 + xy = x^3 - t^d$$

where $d = p^n + 1$ and n is a positive integer. Then $j(E) \notin \mathbb{F}_p$, the conjecture of Birch and Swinnerton-Dyer holds for E over K, and the rank of $E(K)$ is at least $(p^n - 1)/2n$.

By *the Shioda-Tate formula* and assuming maximal Picard number of $\rho = 10n$ for Faltings height n (while $b_2 = 12n - 2$), we know that $r = 10n - rk(T)$ where T is the trivial lattice. Ulmer's proof shows that as the height of Ulmer's curve goes up as $n = 1 + \left\lfloor \frac{d-1}{6} \right\rfloor \to \infty$, the algebraic/analytic rank r goes up to ∞.

Sketch of Ulmer's proof

1. Construct an elliptic surface $S \to \mathbb{P}^1$ over $\mathbb{F}_p$ with generic fiber $E : y^2 + xy = x^3 - t^d$ for $d = p^n + 1$ and $n \in \mathbb{Z}_+$.

2. Construct (and carefully study) a birational isomorphism between S and F_d/G, the quotient of a Fermat surface i.e. $V(x^d + y^d + z^d + w^d) \subset \mathbb{P}^3$ ($d = 4$ then it is K3 surface).

3. Using the fact that the Tate conjecture for surfaces is known for Fermat surfaces, one can deduce the Tate conjecture for S.

4. Use the fact that the Tate conjecture for S implies the Birch and Swinnerton-Dyer conjecture for E. Thus the ranks of the elliptic curves in the family all equal their analytic ranks.

5. The analytic ranks can be computed by relating the L-function of E to the zeta function of S, which can be related to the zeta function of F_d, which is known by Gauss sum computation of Weil. From this one is able to compute the analytic rank which is unbounded from below.

General Global Function Field Case

Theorem (Dori Bejleri–Tristan Phillips–Matthew Satriano–Park; April 2025)

Let $n \in \mathbb{Z}_{\geq 2}$ and $\mathrm{char}(k) \neq 2, 3$. Consider following moduli stacks

- $\mathcal{W}_n^{\min}$ of minimal elliptic fibrations over C_k of height n
- $\mathcal{W}_n^{\Theta}$ of minimal elliptic fibrations over C_k of height n having exactly one specified singular fiber of Kodaira type Θ at a (varying) degree-one place and semistable everywhere else.

Their respective weighted point counts satisfy asymptotically

$$\lim_{B \to \infty} \frac{\mathcal{N}^{\Theta}(\mathbb{F}_q(C),\ B)}{\mathcal{N}^{min}(\mathbb{F}_q(C),\ B)} = |C(\mathbb{F}_q)| \frac{\zeta_C(10)}{\zeta_C(2)} \cdot \frac{q^2}{q^2 - 1} \cdot \kappa(\Theta_u)$$

where $\kappa(\Theta_u)$ is an explicit ratio in q depending only on type Θ

Reduction type Θ with $j \in \overline{M}_{1,1}$	$\kappa(\Theta_v)$
$\mathrm{I}_{k>0}$ with $j = \infty$	$\frac{q-1}{q^2}$
II with $j = 0$	$\frac{q-1}{q^3}$
III with $j = 1728$	$\frac{q-1}{q^4}$
IV with $j = 0$	$\frac{q-1}{q^5}$
$\mathrm{I}_{k>0}^*$ with $j = \infty$ I_0^* with $j \neq 0, 1728$	$\frac{q-1}{q^7}$
I_0^* with $j = 0, 1728$	$\frac{q-1}{q^6}$
IV^* with $j = 0$	$\frac{q-1}{q^8}$
III^* with $j = 1728$	$\frac{q-1}{q^9}$
II^* with $j = 0$	$\frac{q-1}{q^{10}}$

We could specialize to the $K = \mathbb{F}_q(t)$ case where we know the exact values of $|C(\mathbb{F}_q)|\frac{\zeta_C(10)}{\zeta_C(2)}$ by $|\mathbb{P}^1(\mathbb{F}_q)| = q + 1$ and $\zeta_{\mathbb{P}^1_{\mathbb{F}_q}}(s) = 1/(1 - q^{-s})(1 - q \cdot q^{-s})$.