ISSN 0081-5438, Proceedings of the Steklov Institute of Mathematics, 2019, Vol. 307, pp. 1–21. © Pleiades Publishing, Ltd., 2019.
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Igor Rostislavovich Shafarevich and His Mathematical Heritage

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Received November 9, 2019; revised November 20, 2019; accepted November 28, 2019

Abstract—A brief biography of Igor Rostislavovich Shafarevich and an overview of all his main scientific publications are presented.

DOI: 10.1134/S0081543819060014

1. ON IGOR ROSTISLAVOVICH SHAFAREVICH

The contribution of I. R. Shafarevich to Russian and world mathematics can hardly be overestimated. This contribution is measured not only by his personal results and the problems stated by him in algebra, number theory, and algebraic geometry, but also by the huge scientific school that he built, influencing young people over the years through academic advising, seminars, university lectures, and books.

I. R. Shafarevich was born on June 3, 1923, in Zhitomir. His aptitude for mathematics manifested itself very early. In 1938, after finishing grade 9 of secondary school, he entered the Faculty of Mechanics and Mathematics, Moscow State University (MSU), without passing examinations. After two years, he successfully graduated from the university and, at the age of 17, entered postgraduate courses under the supervision of B. N. Delone (Delaunay).

After defending his candidate's dissertation in 1942, Shafarevich became a lecturer at MSU, and in 1946 he defended his doctoral dissertation and became a research fellow at the Steklov Mathematical Institute. Meanwhile, he continued teaching at MSU until 1975, when he was dismissed from MSU because of his public activity. Shafarevich was forced to move his seminar to the Steklov Institute, where it is still active and continues to attract a large number of participants.

From 1960 to 1995, Shafarevich headed the Department of Algebra at the Steklov Mathematical Institute, and from 1995 he was a chief scientific researcher and councilor of the Russian Academy of Sciences. From 1952 to 2017, Shafarevich was a member of the editorial board of the journal *Izvestiya RAN, Seriya Matematicheskaya* (*Izvestiya: Mathematics*), holding the position of deputy editor-in-chief of this journal from 1957 to 1977. From 1970 to 1973, Shafarevich was President of the Moscow Mathematical Society.

Shafarevich's scientific contributions gained wide national and international recognition. In 1958 he was elected a corresponding member of the USSR Academy of Sciences, and in 1991, a full member of the Russian Academy of Sciences. He was also a member of the US National Academy of Sciences, the American Academy of Arts and Sciences, the Royal Society of London, the German National Academy of Sciences Leopoldina, and a foreign member of the Accademia dei Lincei (Italy).

Shafarevich was a Lenin Prize winner; he was awarded the Dannie-Heineman Prize from the Göttingen Academy of Sciences and Humanities and granted an Honorary Doctor degree by the University of Paris–XI (Orsay).

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In 1962 Shafarevich was an invited plenary speaker at the International Congress of Mathematicians in Stockholm, and in 1970 he was an invited speaker in the algebra section at the International Congress of Mathematicians in Nice.

Shafarevich wrote many monographs and textbooks (partly co-authored) based on the courses he taught. They have become a part of the "golden fund" of mathematical literature. The lucidity and clarity of exposition, abundance of informal examples and motivations, and a gradual transition from simple to more complicated situations are characteristic features of the books by Shafarevich.

The book *Number Theory* [19], written jointly with Z. I. Borevich, is an unrivaled work based on long-term lecture courses delivered by Shafarevich at Moscow State University. This book was immediately translated into English, German, French, and Japanese. The book *Basic Algebraic Geometry* [138]—one of the best textbooks on algebraic geometry in world literature—also won wide popularity. A multifaceted exposition of the theory of zeta functions was given in the lectures [136]. The book [84], written jointly with V. V. Nikulin, provides an introduction to modern geometry and is accessible even to senior schoolchildren.

From the very beginning, Shafarevich took an active part in the publication of *Encyclopaedia of Mathematical Sciences*, which started to come out in the 1980s thanks to the efforts of R.V. Gamkrelidze. Shafarevich edited issues in algebra, number theory, and algebraic geometry and had a decisive influence on the content and style of the reviews published in these issues. Almost 80 issues of this edition provide an overview of a considerable part of modern mathematics, and the role of Shafarevich in their appearance is significant. The survey of the basic notions of algebra [140], written in one breath, immediately became widely known, and not only among mathematicians. The overview of the theory of algebraic surfaces [49], written jointly with V. A. Iskovskikh, became a fine textbook for students interested in algebraic geometry.

Besides 35 direct students of I. R. Shafarevich—algebraists, geometers, and specialists in number theory—Shafarevich's school comprises more than 300 researchers, the overwhelming majority of which are worldwide known mathematicians of the highest level. Each of the many students of Shafarevich can regard the time spent with him as the happiest period of their creative development.

Shafarevich is the author of more than 50 original papers. Below we give an overview of all his main scientific publications. Note that Shafarevich's results have also been described, in more or less detail, in many other review papers, some of which we partially used (see [27, 59, 65, 32, 156, 88]).

2. REVIEW OF RESULTS

1. Valuation fields (1943). In his first paper [118], Shafarevich characterized topological fields in which the topology can be defined in terms of valuation. The criterion is formulated in terms of the notions of topologically nilpotent elements and bounded sets, which are actively used, in particular, in the modern theory of Huber rings.

This paper, published in 1943, was noticed by the mathematical community: in the 1948 paper [50], the famous Canadian–American mathematician I. Kaplansky emphasized the elegant result obtained by Shafarevich.

After defending his candidate's dissertation on this subject, Shafarevich did not return to it any more.

2. Extensions of local fields (1946, 1947). For a whole decade, Shafarevich focused his attention on Galois theory and algebraic number theory. Among other problems, he tackled the inverse problem of Galois theory and the embedding problem for local and global fields. Recall that given a field K and a finite group G, the inverse problem of Galois theory consists in finding a normal extension $K \subset L$ with Galois group G. In the embedding problem, one is additionally given a surjective homomorphism $G \to H$ and a normal extension $K \subset E$ with Galois group H, and it is required to find an extension $K \subset L$ as above and an embedding $E \subset L$ over K that corresponds to the surjection $G \to H$ according to the Galois theory. Below we will use this notation in the context of the inverse problem of Galois theory and the embedding problem.

The first significant achievements of Shafarevich in these fields were presented in [119, 120], where he studied extensions of local fields. In [119], for a finite normal extension of local fields $K \subset L$, he demonstrated the relationship between the Brauer group $\mathrm{Br}(L/K)$ and an extension of the Galois group $\mathrm{Gal}(L/K)$ by the abelian quotient of the absolute Galois group of L. This simple but fundamental observation, which was later made independently by many other mathematicians, is now a standard basic fact of the local class field theory.

In [120], Shafarevich proved that for a local field K of characteristic zero that does not contain p-primary roots of unity, the Galois group of the maximal p-extension of K is a free pro-p-group with $[K:\mathbb{Q}_p]+1$ generators. As an application, he solved the inverse problem of Galois theory and the embedding problem for such fields and finite p-groups.

The paper [120] was awarded the prize of the Moscow Mathematical Society. In addition, the papers [119, 120] formed the content of Shafarevich's doctoral dissertation. It is interesting to note that Shafarevich obtained the results of [120] without explicitly using group cohomology theory, which then just appeared. Anticipating thus the emergence of homological algebra in his studies, Shafarevich subsequently made it one of his main mathematical tools.

3. General reciprocity law (1950). The paper [121] is one of the greatest achievements in algebraic number theory. Shafarevich was the first to give in [121] a local formula for a ramified norm residue symbol in terms of a construction that does not go beyond the framework of the local field under consideration. More precisely, for a local filed K of characteristic zero with residue field characteristic equal to $p > 2$ that contains the group μ_p of pth roots of unity and for arbitrary nonzero elements $\alpha, \beta \in K^*$, Shafarevich found an explicit formula for the norm residue symbol $(\alpha, \beta) \in \mu_p$ in terms of the decomposition of the elements on a special basis of the multiplicative group of K, which is now called the Shafarevich basis. Recall that in the implicit form, the symbol (α, β) is defined as the image of α under the local Artin map $K^* \to \mathrm{Gal}(L/K) \simeq \mu_p$, where $L = K(\sqrt[p]{\beta})$, which requires going beyond the field K.

The most important conceptual distinctive feature of Shafarevich's formula is its clearly demonstrated analogy with the residue of a differential form $\alpha\, d\beta$ on a Riemann surface. The analogy between numbers and functions became an integral part of all scientific work of Shafarevich, was conveyed to his students, and became one of the main ideas of the Moscow school of algebraic geometry.

Applying the explicit formula found by him as well as a result of Hasse in class field theory, Shafarevich obtained a general reciprocity law. Thus he solved the problem associated with the names of many famous mathematicians, including Gauss, Jacobi, Kummer, Hilbert, and others. This problem was included by Hilbert among his famous problems (Hilbert's ninth problem). This reciprocity law generalizes the Gauss quadratic reciprocity law, which states that the equality $\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{(a-1)(b-1)/2}$ holds for the Kronecker–Jacobi symbols of odd positive integers a and b, to the case of an arbitrary number field containing μ_p and of μ_p-valued pth power residue symbols. Note that Shafarevich's formula is an analog for $p > 2$ of the right-hand side of the quadratic reciprocity law.

Developing Shafarevich's results, S. V. Vostokov [170] found an explicit formula that expresses the norm residue symbol in terms of power series expansions of α and β in the local uniformizing parameter. This formula was also independently obtained by H. Brückner [21].

Note also that the explicit local formula allows one to more naturally construct both local and global class field theory. This was the subject of papers [70, 71] by A. I. Lapin, the first student of Shafarevich.

Later, far-reaching generalizations of Shafarevich's explicit formula were obtained by many mathematicians, and various interpretations were given, including those in terms of modern p-adic cohomology theories.

4. Inverse problem of Galois theory (1954). After his work on the general reciprocity law, Shafarevich turned again to the inverse problem of Galois theory, but now for global rather than local number fields. His efforts were crowned with success, and he published a large series of papers on this subject in 1954.

First, in [122], he found a new method for solving the inverse problem of Galois theory for algebraic number fields and l-groups with l an arbitrary prime number (see also the comments in the collected works [149, p. 752]). This problem was solved earlier by A. Scholz [116] and H. Reichardt [101] for $l > 2$; however, Shafarevich's method allows one to construct much more various extensions with a given Galois group than the Scholz–Reichardt method.

In [123], the method developed in [122] led to the solution of the embedding problem in the following case: Shafarevich considered a finite normal extension of number fields $K \subset E$ with Galois group H and a surjective homomorphism of finite groups $G \to H$, for which he assumed that it has a group section, that its kernel N is an l-group, and that either l does not divide the order of H or N is a nilpotent group of nilpotency class less than l. Earlier, a similar result was known only in the case where the kernel N is abelian (this was proved by Scholz [115]).

Note that subsequently the technique from the embedding problem for number fields was repeatedly used by Shafarevich as applied to number fields and arithmetic geometry (see Subsections 7 and 9 below).

Finally, in [124], Shafarevich obtained a fundamental result stating that for any number field K and a solvable group G, there exists a normal extension $K \subset L$ with Galois group G. Moreover, it turns out that there are infinitely many such extensions.

The technique used in deriving these results involves fine arithmetic properties of number fields. For instance, the effective construction of fields with a given solvable Galois group is split into steps, and the consistency of the results of each step requires truly exquisite workmanship of every detail of the construction. Moreover, in these studies Shafarevich started to apply homological algebra in an essential way, using a number of D. K. Faddeev's results on group homology.

For the series of articles on the solution to the inverse problem of Galois theory for algebraic number fields and solvable groups, Shafarevich was awarded the Lenin Prize in 1959.

The inverse problem of Galois theory for algebraic number fields and unsolvable groups was the subject of later papers by Shafarevich's student G. V. Belyi. In [12], Belyi solved this problem for the maximal abelian extension of the field $\mathbb{Q}$ and for most of the classical Chevalley groups over finite fields. As an auxiliary result, Belyi characterized complex smooth projective curves that can be defined over an algebraic number field as curves covering the projective line with ramification only over three points. This result became widely known and found numerous applications, as well as became a starting point for the well-known Grothendieck's program of dessins d'enfants [158] devoted to the visualization of the Galois group of the field of rationals through its action on the graphs of some special type embedded in smooth oriented compact surfaces. This direction was extensively developed in the studies of G. B. Shabat, another student of Shafarevich (see, for example, the survey [114]).

5. Embedding problem for local and global fields (1958, 1959, 1962). In the short paper [128], using the methods developed in his previous studies, Shafarevich solved the embedding problem for number fields in the case where G is the semidirect product of a group H and a nilpotent group N.

Next, in [28], I. R. Shafarevich and S. P. Demushkin (one of his early students) solved the embedding problem for local fields in the case of an abelian group N. Earlier, R. Brauer [20],

B. N. Delone and D. K. Faddeev [26], and H. Hasse [44] introduced the following obstruction to the solution of the embedding problem. Every group homomorphism $\chi\colon N \to E^*$ to the multiplicative group of nonzero elements of the field E (see the notation in the beginning of Subsection 2) defines a class $\chi(\alpha) \in H^2(H_\chi, E^*)$, where $\alpha \in H^2(H, N)$ is a class of a given extension of the groups and H_χ is the stabilizer of the character χ in the group H. If there is an embedding $H^2(H_\chi, E^*) \simeq \mathrm{Br}(E/K) \subset \mathrm{Br}(K)$ and the embedding problem has a solution, then one can easily see that $\chi(\alpha) = 0$. The set of conditions $\chi(\alpha) = 0$ over all characters χ is called the first obstruction to solving the embedding problem. In [28], Shafarevich and Demushkin proved that if K is a local field of characteristic zero and the group N is abelian, then the triviality of the first obstruction is equivalent to the existence of a solution to the embedding problem.

In [29], they investigated the so-called second obstruction to solving the embedding problem for algebraic number fields and, in some particular cases, solved the embedding problem.

6. Cohomology of nilpotent algebras (1957). In the 1950s, in many fields of mathematics, there was a clear feeling of the importance of the methods of homological algebra, which started not only to find applications in these fields but also to determine the very directions of research. In particular, this fact was explicitly pointed out in Shafarevich's notes about the International Congress of Mathematicians in Edinburgh [130].

In the note [60], written during the period of expansion of the methods of homological algebra, Shafarevich and his student A. I. Kostrikin considered a finite-dimensional associative unital algebra A over a field k with augmentation $A \to k$ that has a nilpotent kernel. They studied the behavior of the dimensions of the cohomology groups $H^i(A, k) := \mathrm{Ext}^i_A(k, k)$; namely, for the Poincaré series $P_A(t) = \sum_{i\ge 0} \dim H^i(A, k)t^i$, they obtained lower and upper bounds for its coefficients and raised the question of whether this series is rational.

Although it was found out that in the general case the series $P_A(t)$ may be irrational (see, for example, the survey [8]), in a particular case important from the arithmetic viewpoint, when $k = \mathbb{F}_p$ and A is the group algebra $\mathbb{F}_p[G]$ of a finite p-group G, the series $P_A(t)$ turns out to be rational, which was soon proved by Shafarevich's student E. S. Golod [39].

Similar questions on the generating functions of graded objects arose from about 1950 to 1970 for different reasons in various fields of mathematics.

7. Principal homogeneous spaces (1957, 1959). In the mid-1950s, Shafarevich was engaged in algebraic geometry. Naturally, the first problems that he started to consider here were at the interface between number theory and geometry. Namely, they pertained to the theory of elliptic curves.

The first ideas in this field were expressed in Shafarevich's report at the III All-Union Mathematical Congress [125]. Here he pointed out the analogy between the embedding problem in the Galois theory of algebraic number fields and the problem of classifying genus 1 curves defined over algebraic number fields. The objects studied in both theories have local invariants related to completions of the field of definition, and the main interest lies precisely in "locally trivial" objects.

In [126], Shafarevich established a bijection between the set of genus 1 curves over a field k whose Jacobian is isomorphic to a given elliptic curve E over k and the Galois cohomology group $H^1(G_k, E(k^{\mathrm{sep}}))$, where $E(k^{\mathrm{sep}})$ is the group of points of E over a separable closure k^{sep} of k and $G_k = \mathrm{Gal}(k^{\mathrm{sep}}/k)$ is the absolute Galois group of k. Earlier, it was known from F. Châtelet's paper [23] that this set of curves forms a group; later this fact was generalized to abelian varieties by A. Weil [172]. However, a cohomological interpretation first appeared precisely in [126] and eventually found far-reaching generalizations, becoming a standard tool in algebraic and arithmetic geometry.

Using this cohomological interpretation as well as results related to the embedding problem for number fields, Shafarevich solved in [127] the following long-standing problem in the theory of

Diophantine equations: prove that for an arbitrary positive integer d, there exists a curve of genus 1 over the field of rationals that can be smoothly embedded in the projective space as a curve of degree d and cannot be embedded as a curve of lower degree. All these results were the first steps in the new branch of algebraic geometry, the theory of principal homogeneous spaces, which emerged in the 1950s and was actively developed by Shafarevich as well as independently by S. Lang and J. Tate [69].

Then, in [129], Shafarevich generalized the main result of [127] to arbitrary abelian varieties over number fields. For an arbitrary abelian variety A over a local field k, he constructed in [129] a nondegenerate pairing between $H^1(G_k, A(k^{\mathrm{sep}}))$ and $\widehat{A}(k)$, where $\widehat{A}$ is the dual abelian variety of A. In addition, in this short note Shafarevich introduced the kernel of the natural localization homomorphism, which consists of classes of homogeneous spaces over a number field that have a point in any of its completions. In honor of the author, this group is denoted in the world mathematical literature by the Russian character Ш and is called the Shafarevich–Tate group. In [129], Shafarevich proved that for any positive integer n the n-torsion subgroup in the Shafarevich–Tate group is finite. The calculation of this group and, in particular, the proof of the conjectured finiteness are the most difficult and interesting problems in the theory of Diophantine equations. For several decades, the finiteness of the group Ш could not been proved for any abelian variety. The first example of an elliptic curve over a number field with a finite Shafarevich–Tate group was constructed by K. Rubin [103]. The strongest results were obtained here by Shafarevich's student V. A. Kolyvagin [55], who proved the finiteness of the Shafarevich–Tate group for all (modular) elliptic curves with analytic rank at most 1; he also proved the Birch and Swinnerton-Dyer conjecture for such elliptic curves. Note that for an elliptic curve E over a local field k, Shafarevich constructed in [129] a nondegenerate pairing between $H^1(G_k, E(k^{\mathrm{sep}}))$ and $E(k)$.

In the extensive study [131], Shafarevich carried out a fundamental investigation of principal homogeneous spaces with respect to an abelian variety A over the field of rational functions K on a curve C over an algebraically closed field k of arbitrary characteristic $p \geq 0$; namely, he studied the group ${}_pH^1(G_K, A(K^{\mathrm{sep}}))$. Here the subscript p means that one considers the product of l-primary components of the abelian group, with l running through all prime numbers different from p (for $p = 0$, this condition is empty). Shafarevich first considered the local case: for a complete discrete valuation field L with algebraically closed residue field k (the characteristic of L may be zero), he proved that the group ${}_pH^1(G_L, A(L^{\mathrm{sep}}))$ is the Pontryagin dual of the group $\pi_1(\widehat{A}(L))$, where $\widehat{A}(L)$ is viewed as a proalgebraic group over k. In more explicit terms, the group $\pi_1(\widehat{A}(L))$ is isomorphic to the Tate module ${}_pT(\widehat{A}(L))$. Shafarevich conjectured that there is a similar duality for the p-primary part as well; this conjecture was later proved by his student O. N. Vvedenskii in the case of elliptic curves [171] and then in [11, 14, 13] in the general case.

Next, in [131], Shafarevich described in the global case the kernel and cokernel of the homomorphism

$${}_pH^1\big(G_K, A(K^{\mathrm{sep}})\big) \to \bigoplus_{x\in C} {}_pH^1\big(G_{K_x}, A(K_x^{\mathrm{sep}})\big),$$

where K_x is the completion of the field K with respect to the valuation defined by a point $x \in C$. Before the emergence of étale cohomology theory, Shafarevich actually calculated the Euler characteristic of a constructible sheaf $\mathcal{A}_n$ in the étale topology on C, where $\mathcal{A}$ is the Néron model over C of an abelian variety A over K and n is a positive integer coprime to p. Later, the reduction of étale cohomology to Galois cohomology became a standard technique. Similar results were obtained independently by A. P. Ogg [89] and then generalized by A. Grothendieck [43]. The corresponding formula for the Euler characteristic of an étale sheaf on a curve was called the Grothendieck–Ogg–Shafarevich formula. In addition to numerous applications to constant and unramified abelian varieties over K, in [131] Shafarevich introduced a criterion for an abelian variety to be unramified in terms of a representation of the Galois group, which was later called the Néron–Ogg–Shafarevich criterion.

8. Conjectures on curves with given points of bad reduction (1963). One characteristic feature of Shafarevich's further research can be seen already in his work on principal homogeneous spaces: in most of his studies he treats geometry as a number theorist and, vice versa, number theory as a geometer.

The plenary talk by Shafarevich at the 1962 International Congress of Mathematicians in Stockholm [132] was organized in precisely this way. In addition to the results on the extensions of number fields (see Subsection 9 below), a special focus in this report was placed on two conjectures on algebraic curves defined over a global field, i.e., over a number field or over the field of rational functions on a curve over a finite field. For every curve X over a global field K, a finite set S of non-Archimedean points of K is defined at which X has bad reduction. Inspired by the classical theorems of C. Hermite and H. Minkowski, Shafarevich conjectured that the number of curves over K with given genus $g \geq 2$ and set S is finite. Moreover, he conjectured that for $K = \mathbb{Q}$, $g \geq 1$, and $S = \varnothing$ there are no such curves at all. In the functional case, the latter conjecture can also be formulated for the field $K = \mathbb{F}_q(t)$ if one rules out isotrivial families of curves.

In [94], Shafarevich's student A. N. Parshin considered analogs of these questions for curves over the field $\mathbb{C}(B)$ of rational functions on a complex smooth projective curve B. Every such curve X of genus $g \geq 2$ can be assigned a point P_X with coordinates in the field $\mathbb{C}(B)$ on the projective space $\mathbb{P}^n$, and the curve X is uniquely recovered by the point P_X. Parshin proved that the height of the points P_X is uniformly bounded over all curves X of fixed genus $g \geq 2$ with fixed set of points of bad reduction $S \subset B$. This implies that all such curves are defined by complex points of some complex algebraic variety. Along with these results, Parshin proved in [94] the absence of deformations for nonisotrivial families of curves over B with $S = \varnothing$, which implied the finiteness conjecture in this case. Then S. Yu. Arakelov (another student of Shafarevich) showed [4] that nonisotrivial families do not admit deformations in the general case either, and thus proved the finiteness conjecture for arbitrary curves X of genus $g \geq 2$ over the field $\mathbb{C}(B)$.

In addition, Parshin found in [94] that Shafarevich's finiteness conjecture implies Mordell's conjecture on the finiteness of the number of rational points on curves of genus $g \geq 2$; the relevant argument was subsequently called Parshin's trick. In particular, along with the above arguments, this trick allowed Parshin to prove in [94] an analog of Mordell's conjecture for curves over $\mathbb{C}(B)$.

Note that this analog of Mordell's conjecture was proved earlier by another student of Shafarevich, Yu. I. Manin [74], in 1963, and independently by H. Grauert [42] in 1965. A notion used by Manin in his proof is called since then the Gauss–Manin connection; it has become one of the standard tools in the study of families of complex algebraic varieties.

Parshin's trick as well as a result of Yu. G. Zarkhin [175]—another representative of Shafarevich's school (Manin's student)—were the most important steps toward the proof of Shafarevich's finiteness conjecture and, hence, Mordell's conjecture on number fields. Such a proof was found by G. Faltings [35].

In the general case, the problem of extending Shafarevich's finiteness conjecture to families of algebraic varieties of higher dimension remains open. For surfaces of general type over the field $\mathbb{C}(B)$, the corresponding finiteness result was proved by E. Bedulev (Parshin's student) and E. Viehweg [10].

Shafarevich's conjecture on the absence of abelian varieties without bad reduction over $\mathbb{Q}$ was proved later by his student V. A. Abrashkin [1–3] and independently by J.-M. Fontaine [37]. They also demonstrated that abelian varieties without bad reduction do not exist over some extensions of the field $\mathbb{Q}$ either; in particular, they do not exist over the quadratic fields $\mathbb{Q}(\sqrt{d})$ for $d = -1, -2, -3, -7, 2, 5$.

9. Extensions of number fields with given ramification points (1963). Shafarevich returned to the study of the Galois groups of number fields in his important paper [133] published in the journal *Publications mathématiques de l'IHÉS* in Russian. Here he considered p-extensions of an algebraic number field K that are ramified in a fixed finite set S of non-Archimedean points of K. This can be viewed as a zero-dimensional variant of the problem of describing curves over K with a bad reduction set S (see Subsection 8).

Let $G_K(p, S)$ be the Galois group of the maximal extension of a field K with the above properties. In [133], using class field theory, Shafarevich obtained an explicit formula for the minimum number d of generators of $G_K(p, S)$ in terms of arithmetic invariants of the field K and the set S. The main result of this paper is an upper bound for the number of relations r in the minimal set of generators of the group $G_K(p, S)$. Here Shafarevich applied the elegant technique from the solution of the embedding problem for number fields. In particular, if p is coprime to all points in S, then $r \le d + \rho$, where ρ is the number of generators of the group of units of the field K; therefore, in this case, the number $r - d$ is uniformly bounded over all number fields K of fixed degree over $\mathbb{Q}$. In addition, using a general estimate for the number r, Shafarevich obtained an exact description of the group $G_K(p, S)$ in a number of nontrivial examples.

Moreover, in [133], Shafarevich noticed that a uniform bound for the number of generators would imply a solution to the well-known tower problem in class field theory if one could prove the following group-theoretic statement: there exists a strictly increasing function $\rho(d)$ such that for any finite p-group G one has the inequality $r > d + \rho(d)$ where d is the minimum number of generators of G and r is the minimum number of relations between them. In other words, this statement means that the number of relations of a finite p-group is necessarily large enough compared with the number of generators.

All this was discussed in detail in Shafarevich's talk at the International Congress of Mathematicians in Stockholm [132].

Later, Galois groups of the form $G_K(p, S)$ were thoroughly studied by Shafarevich's student H. Koch [52].

10. Class field towers (1964). In [41], I. R. Shafarevich and his student E. S. Golod proved that the inequality $r > (d-1)^2/4$ holds for any finite p-group G, where d is the minimum number of generators of G and r is the minimum number of relations between them. Here they developed methods of homological algebra and applied them to the study of noncommutative rings. Note that later a modification of the Golod–Shafarevich approach allowed E. B. Vinberg [168] and independently P. Roquette [102] to obtain the stronger inequality $r > d^2/4$.

According to what has been said in Subsection 9, the main result of [41] gave a solution to the class field tower problem. This problem resisted the efforts of specialists in algebraic number theory and algebra for more than 40 years; moreover, it was not even clear whether the answer should be positive or negative. Namely, as a consequence of the above group-theoretic result, Shafarevich and Golod showed that if the number of generators of the p-part of the class group of K is large enough compared with its degree, then the field K has an infinite unramified prosolvable extension. In particular, this is so for $p = 2$ for any imaginary quadratic field whose discriminant has at least six prime divisors.

The solution obtained and the technique used in its proof have many consequences in number theory and algebra. It suffices to mention the proof of the existence of algebraic number fields that cannot be embedded in one-class fields, sharp estimates for the growth of the discriminant of a number field depending on its degree, as well as Golod's negative solution to the generalized Burnside problem [40], which consists in constructing an infinite residually finite finitely generated group whose every element has p-primary order (without uniform boundedness of these orders). Note that a positive solution to the weakened Burnside problem for a prime exponent p was obtained earlier by

Kostrikin [56, 58]; namely, Kostrikin proved that for every positive integer n there exist only finitely many finite groups of exponent p that are generated by n elements. The solution to the weakened Burnside problem for an arbitrary exponent was later obtained by E. I. Zel'manov [176, 177]. Note that the negative solution to the classical Burnside problem with uniformly bounded orders of elements was obtained by P. S. Novikov and S. I. Adyan in the series of works [85–87], where they constructed examples of infinite finitely generated groups of finite exponent. Moreover, the results and methods of [41] found unexpected applications in topology in the study of fundamental groups of hyperbolic three-dimensional smooth manifolds (see the papers by A. Lubotzky [73] and M. Lackenby [67, 68]).

The response to [41] turned out to be so significant that its main result almost immediately appeared in monographs and academic books. It is surprising that such a natural problem of combinatorial group theory, considered separately without any relation to algebraic number fields, was not solved earlier, although other questions closely related to it were studied for a long time. In particular, since the time of J. Schur [117], the question of the possible number d of generators for finite p-groups satisfying the condition $r = d$ was repeatedly raised.

Later, Shafarevich returned to problems related to number fields at various times. In the course of lectures [136] delivered at the Faculty of Mechanics and Mathematics, MSU, in 1966–1967, which were recorded by Manin, Shafarevich presented for the first time the results of K. Iwasawa, not yet fully published at the time, on an analog of the Tate module for algebraic number fields and on the related problem of the p-adic extension of L-functions of number fields. Later, Manin [76, 77] developed the theory of such L-functions and their generalization. These questions were also addressed by L. V. Kuz'min [66] (Golod's student).

11. Algebraic surfaces (1965). In 1961–1963, Shafarevich organized and headed a small seminar, mainly for his students. Among the members were B. G. Averbukh, Yu. I. Manin, B. G. Moishezon, A. N. Tyurin, G. N. Tyurina, Yu. R. Vainberg, and A. B. Zhizhchenko. The aim of the seminar was to understand some classical results of Italian algebraic geometers from the modern (at the time) point of view. Similar activity occurred independently in USA due to O. Zariski and D. Mumford and in Japan due to K. Kodaira.

The work of the seminar headed by Shafarevich resulted in the monograph [150], which remained for a long time the only systematic exposition of the theory of surfaces; this monograph combined the elegance of the classical geometric methods of the Italian school with the up-to-date analytic, topological, and algebraic methods, including the Hodge theory and the cohomology theory of algebraic coherent sheaves developed by J.-P. Serre. The book provided a full classification of algebraic surfaces, as was done by Italians; in some places, the classical statements were corrected or augmented. This work, primarily attributed to the efforts of Shafarevich, had an enormous impact on the study of algebraic surfaces throughout the world; it was translated into English in 1967 and into German in 1968.

The contribution of Shafarevich to the theory of algebraic surfaces and, specifically, to the monograph [150] significantly exceeds what can be reconstructed from the two chapters of this book written personally by him. In one of the chapters, Shafarevich gave a modern proof of the Enriques criterion of linearity of algebraic surfaces. The other chapter describes surfaces with an elliptic pencil in terms of principal homogeneous spaces; moreover, it turned out that a whole infinite series of such surfaces (with multiple fibers) was missed in the classical literature.

It is remarkable that the seminar and the book served as the main impetus for the further development of algebraic geometry in Moscow. The investigations stemming directly from this work include the following: rational surfaces and rational multidimensional varieties, including the solution to Lüroth's problem (V. A. Iskovskikh and Yu. I. Manin [48]) and the classification of three-dimensional Fano varieties (V. A. Iskovskikh [46, 47]); the theory of vector bundles on algebraic curves and surfaces (A. N. Tyurin [162, 163] and F. A. Bogomolov [18]); K3 surfaces (G. N. Tyurina [165], I. R. Shafarevich and I. I. Piatetski-Shapiro [99, 100], V. V. Nikulin [81–83], A. N. Rudakov, Vik. S. Kulikov [63, 64], and others; see Subsections 17 and 18 below); multidimensional birational and analytic geometry, including the abundance criterion (B. G. Moishezon [79, 80]); minimal models of arithmetic surfaces (I. R. Shafarevich [134]; see Subsection 12 below); irrational simply connected surfaces with trivial geometric genus (I. V. Dolgachev [31]); geometry and arithmetic of rational surfaces (Yu. I. Manin [75]); and classification of complex nonalgebraic surfaces (F. A. Bogomolov [17]).

12. Arithmetic surfaces (1966). The comparison of numerical and geometric situations, which was brilliantly used by Shafarevich already in his work on the general reciprocity law (see Subsection 3), is a long-standing tradition in number theory and algebraic geometry. It goes back to L. Kronecker and D. Hilbert. In line with these ideas, in his lectures delivered in Bombay [134], Shafarevich developed intersection theory and constructions of minimal models and the canonical class for schemes of dimension 2: both for ordinary two-dimensional algebraic varieties over classical fields and for algebraic curves over natural rings such as the ring of integers. This unified exposition sheds new light on both geometric and arithmetic cases. Actually, [134] was the first publication showing that an arithmetic surface can be fully analyzed as a geometric object; many subsequent strong results in number theory would have been inconceivable without this work.

In his lectures [134], Shafarevich raised the problem of constructing intersection theory and a canonical class on an arithmetic surface that would involve its Archimedean fibers. Without considering these fibers, the surface is not complete, and, by analogy with the geometric case, one cannot hope for the existence of intersection theory.

This program was implemented by S. Yu. Arakelov [5, 6]. The starting point for constructing intersection theory was the notion of canonical local height, introduced by Parshin [95], which corresponds to the Archimedean components and uses Green's function. Arakelov's contribution initiated a large series of investigations at the interface between algebraic geometry, complex analysis, and number theory, which was called Arakelov geometry (see, for example, the surveys of G. Faltings [36] and C. Soulé with coauthors [155] as well as P. Vojta's paper [169]). In [96], within the framework of Arakelov geometry, Parshin formulated a hypothetical inequality for arithmetic surfaces, which is an analog of the famous Bogomolov–Miyaoka–Yau inequality [18, 78, 173, 174] for algebraic surfaces of general type. In [96], Parshin showed that this arithmetic inequality implies an effective proof of many known conjectures, including Mordell's conjecture, the Szpiro inequality for elliptic curves, and some forms of the abc conjecture.

13. Lie algebras in positive characteristic (1966, 1969). By the mid-1960s, many mathematicians got interested in the classification of simple transitive transformation pseudogroups obtained by E. Cartan. Immediately after a session of the seminar on algebraic surfaces, Shafarevich organized in 1964–1966 a seminar at the Steklov Mathematical Institute in which various works on Lie pseudogroups were discussed. In particular, this discussion resulted in two joint papers [61, 62] with Kostrikin, which determined a classification program for simple finite-dimensional Lie algebras over fields of positive characteristic for further decades.

By that time, a lot of facts had been accumulated in the theory of modular simple Lie algebras. It seemed that the ingenuity displayed in the construction of more and more new examples of simple Lie algebras would never end. The publication of the paper [61] was an important milestone; in this paper, Shafarevich and Kostrikin pointed out that all known nonclassical simple Lie p-algebras, also called bounded Lie algebras, fall for $p > 5$ into four infinite series of Cartan-type algebras W_n, S_n, H_n, and K_n (general, special, Hamiltonian, and contact series, respectively). In [61], Shafarevich and Kostrikin conjectured that Cartan-type algebras along with the classical ones exhaust all finite-dimensional simple Lie p-algebras over an algebraically closed field of characteristic $p > 5$.

In [62], the technique of complete Cartan extensions was developed as applied to arbitrary Lie algebras over a field of characteristic $p > 0$, not necessarily having the structure of a Lie p-algebra. Shafarevich and Kostrikin constructed and studied model examples of simple graded Lie algebras. The results of [62] initiated an extensive classification program, and the paper was cited in almost every study devoted to modular simple Lie algebras. In the talk at the International Congress of Mathematicians in Nice [57], Kostrikin formulated a generalization of the above conjecture to simple (unbounded) Lie algebras.

Numerous subsequent results obtained by many authors agreed well with these conjectures. In the breakthrough paper [16], R. E. Block and R. L. Wilson proved the conjecture on simple Lie p-algebras for $p > 7$. In a long series of papers ending with [154], H. Strade proved a generalized conjecture on simple Lie algebras for $p > 7$. In another series, ending with [97], A. Premet and H. Strade proved similar conjectures for $p = 7$ and also proved a variant of the conjectures for $p = 5$ with the Melikyan algebras added to the classification.

In [104], Shafarevich and his student A. N. Rudakov described all irreducible representations of a three-dimensional Lie algebra over a field of positive characteristic; moreover, they constructed a moduli space of such representations and described some of their degenerations into reducible representations. Note that Shafarevich repeatedly returned to the descriptions of moduli varieties of algebraic objects. The moduli varieties of irreducible representations have later been studied by many other authors in different contexts.

14. Infinite-dimensional groups (1966, 1981, 2004). In [135], one of the few papers written in English, Shafarevich initiated the development of a new direction in algebraic geometry—the theory of infinite-dimensional algebraic varieties—with a view to studying the automorphism group $\mathrm{Aut}(\mathbb{A}^n)$ of the affine space over a field of characteristic zero. Namely, he regarded $\mathrm{Aut}(\mathbb{A}^n)$ as an object of infinite-dimensional algebraic geometry, as a group ind-affine variety in modern terminology. Later, in [139], Shafarevich presented details of the proofs of the main statements from [135].

Shafarevich proved the smoothness of the ind-affine variety $\mathrm{Aut}(\mathbb{A}^n)$. For the subgroup $\mathrm{SAut}(\mathbb{A}^n) \subset \mathrm{Aut}(\mathbb{A}^n)$ consisting of automorphisms with unit Jacobian, he proved that it is simple as a group ind-affine variety. Moreover, he proved that $\mathrm{Aut}(\mathbb{A}^n)$ is generated as a group ind-affine variety by the subgroup of affine transformations and by the subgroup of upper triangular transformations of the form $x_i \mapsto x_i + f_i(x_1, \dots, x_{i-1})$. In the case of $n = 2$, he gave a more detailed description of the group $\mathrm{Aut}(\mathbb{A}^2)$ as a free product and, as an application, proved that the affine plane has no nontrivial forms. For an arbitrary positive integer n, he obtained results on the structure of the Lie algebra of the group $\mathrm{Aut}(\mathbb{A}^n)$ (note that the problem of finding their "integrated" versions is very hard). Shafarevich also remarked that to prove the famous Jacobian conjecture, it suffices to prove that the embedding of $\mathrm{Aut}(\mathbb{A}^n)$ into the semigroup of all endomorphisms of the affine space is closed; however, the latter condition is difficult to verify.

Note that similar results for the group of complex points $\mathrm{Aut}(\mathbb{A}^n)(\mathbb{C})$ on $\mathrm{Aut}(\mathbb{A}^n)$, considered as an abstract group, are not true. Namely, V. I. Danilov [24] proved that $\mathrm{SAut}(\mathbb{A}^n)(\mathbb{C})$ is not a simple group, and I. P. Shestakov and U. U. Umirbaev [151] constructed an example of an element in $\mathrm{Aut}(\mathbb{A}^n)(\mathbb{C})$ that cannot be decomposed into a product of affine and upper triangular transformations.

Later, Shafarevich returned to the study of infinite-dimensional groups in [147], where he investigated the structure of a group ind-affine variety on the group $\mathrm{GL}_2(k[t])$ over a field k.

In recent years, there has been a burst of activity in the study (initiated by Shafarevich) of the automorphism groups of algebraic varieties by the methods of infinite-dimensional geometry (see the survey [38]).

15. Uniformization of algebraic varieties (1966, 1972). Infinite-dimensional objects of algebraic geometry were also considered by Shafarevich in the context of uniformization of algebraic varieties; here, pro- rather than ind-algebraic varieties appeared.

In the joint paper with his student I. I. Piatetski-Shapiro [98], Shafarevich considered an algebraic analog of uniformization. Namely, for an arbitrary variety X, they addressed the question of whether there exists a profinite covering by a pro-algebraic variety $\widetilde{X} \to X$ that is unramified over an open subset in X and is quasi-homogeneous, i.e., such that the general orbit of the group $\mathrm{Aut}(\widetilde{X})$ is dense. They proved that all quotients of bounded symmetric domains by arithmetic groups possess such a property. To this end, they analyzed in detail the fields of automorphic functions and algebras of automorphic forms from an algebraic viewpoint, as well as developed a number of new concepts in the theory of pro-algebraic varieties.

In the context of uniformization of algebraic varieties, Shafarevich also raised the following question, which he formulated at the very end of the textbook on algebraic geometry [138]: Is it true that for any complex smooth projective variety X its simply connected covering $\widetilde{X}$ is holomorphically convex? Recall that the holomorphic convexity of a variety $\widetilde{X}$ is equivalent to the fact that there exists a proper holomorphic map with connected fibers $\widetilde{X} \to S$ to a Stein variety S.

According to J. Kollár, a positive answer to this question is equivalent to the existence of a morphism $X \to \mathrm{Sh}(X)$ of algebraic varieties such that $\mathrm{Sh}(X)$ is a normal variety with connected fibers for which a connected subvariety $Z \subset X$ contracts to a point if and only if the image of the homomorphism $\pi_1(Z) \to \pi_1(X)$ is finite. The morphism $X \to \mathrm{Sh}(X)$ is called Shafarevich's map. If we consider only the maximal abelian quotient of the fundamental groups, then the corresponding variant of Shafarevich's map is given by the Albanese map. Thus, Shafarevich's map can be regarded as a nonabelian variant of the Albanese map. Note that in the popular scientific paper [142] Shafarevich wrote that he was inspired by the philosophy of "nonabelian mathematics of the future" ever since he started to study mathematics.

Although the question of the existence of Shafarevich's map remains open in the general case, significant progress towards its positive solution has been made by many authors, including J. Kollár [53, 54], L. Katzarkov [51], P. Eyssidieux [33], P. Eyssidieux, L. Katzarkov, T. Pantev, and M. Ramachandran [34], R. Treger [160, 161], and others.

16. Rank of elliptic curves (1967). During his stay in Paris in 1966, Shafarevich, together with J. Tate [157], constructed examples of elliptic curves over the field $\mathbb{F}_p(t)$ of arbitrarily large rank. These examples are forms over $\mathbb{F}_p(t)$ of supersingular elliptic curves defined over $\mathbb{F}_p$; in particular, these elliptic curves are isotrivial over $\mathbb{F}_p(t)$.

Examples of nonisotrivial elliptic curves over $\mathbb{F}_p(t)$ with unbounded rank were constructed much later by D. Ulmer [166]. The question of unboundedness of the rank of elliptic curves over number fields remains an open problem in number theory.

17. Torelli's theorem for K3 surfaces and its applications (1971, 1973). One of the striking examples of complex smooth projective surfaces is given by K3 surfaces, which, together with abelian surfaces, are an analog of elliptic curves. It is easy to show that, just as complex elliptic curves, complex abelian surfaces and, more generally, complex abelian varieties are uniquely determined by their first integral Hodge structure. This is associated with the fact that complex abelian varieties can be represented as a quotient of a complex vector space by a free abelian group. The absence of a similar representation for K3 surfaces caused significant problems in attempts to define a K3 surface by its Hodge structure; this problem had long seemed unapproachable to experts. Note that although there is also no such representation for complex smooth projective curves of genus $g \geq 2$, nevertheless, according to Torelli's theorem, these curves can be reconstructed by the first integral Hodge structure together with a natural unimodular pairing on it.

In [99], Shafarevich and Piatetski-Shapiro proved an analog of Torelli's theorem for K3 surfaces; this proof formed the content of Shafarevich's talk at the International Congress of Mathematicians in Nice [137]. Namely, every complex algebraic K3 surface X can be assigned its Hodge structure $H^2(X,\mathbb{Z})$ on which the intersection number defines an integer unimodular pairing. Torelli's theorem for K3 surfaces states that for any two K3 surfaces X and X' there is a naturally defined bijection

$$\mathrm{Isom}(X,X') \xrightarrow{\sim} \mathrm{Isom}^{\mathrm{eff}}\big(H^2(X,\mathbb{Z}),H^2(X',\mathbb{Z})\big),$$

where the right-hand side consists of isomorphisms of Hodge structures that preserve the intersection number and map the classes of effective divisors on X to classes of effective divisors on X'. Note that the Hodge structure on the lattice $H^2(X,\mathbb{Z})$ is uniquely defined by the period vector of a nonzero holomorphic differential 2-form on X.

In the equivalent formulation, this theorem states that there is a naturally defined bijection

$$\mathrm{Isom}\big((X,h),(X',h')\big) \xrightarrow{\sim} \mathrm{Isom}\big((H^2(X,\mathbb{Z}),h),(H^2(X',\mathbb{Z}),h')\big),$$

where the left-hand side consists of isomorphisms of K3 surfaces that map a distinguished ample class $h\in H^2(X,\mathbb{Z})$ to a distinguished ample class $h'\in H^2(X',\mathbb{Z})$, and the right-hand side consists of isomorphisms of Hodge structures that preserve the intersection number and also map h to h'.

As an application of Torelli's theorem for K3 surfaces, Shafarevich and Piatetski-Shapiro described the relationship between the automorphism group of a K3 surface and the arithmetic properties of its lattice of algebraic cycles, which became one of the main methods for the further intense study of the automorphism groups of K3 surfaces. In particular, Shafarevich's student V. V. Nikulin [81–83] classified K3 surfaces with finite automorphism groups.

To prove Torelli's theorem for K3 surfaces, Shafarevich and Piatetski-Shapiro used the local Torelli theorem for K3 surfaces, which was earlier proved by G. N. Tyurina [150, Ch. IX], a detailed study of the periods of Kummer surfaces, as well as some specific families of K3 surfaces constructed with the use of the Hilbert schemes. In fact, these families are thin moduli varieties of (marked) K3 surfaces. The central object of analysis in the proof is the period map from these thin moduli varieties to some explicitly constructed complex symmetric domains. Torelli's theorem was derived from the fact that the period map is an open embedding with dense image.

The subject of Torelli's theorem for K3 surfaces was later developed in many directions. We only mention some of the relevant results. Shafarevich's student Vik. S. Kulikov [63, 64] obtained a classification of semistable degenerations of complex algebraic K3 surfaces, which implies the surjectivity of the period map. Torelli's theorem for Kähler K3 surfaces was proved by D. Burns and M. Rapoport [22] as well as by E. Looijenga and C. Peters [72]. The surjectivity of the period map for Kähler K3 surfaces was proved by Shafarevich's student A. N. Todorov in [159], where he proved in addition that all complex K3 surfaces are Kählerian (see also Y.-T. Siu's paper [152], where the details lacking in [159] are supplied). A. N. Tyurin [164] proved analogs of Torelli's theorem for moduli spaces of vector bundles on algebraic curves. A variant of Torelli's theorem in terms of derived categories was obtained by D. O. Orlov [91]. A generalization of Torelli's theorem to hyper-Käler varieties was found by M. Verbitsky [167].

In the paper [100], which was also written jointly with Piatetski-Shapiro, an analog of the Riemann hypothesis for K3 surfaces over finite fields was derived from Torelli's theorem for complex algebraic K3 surfaces. Note that the proof of this fact given in [100] differs from the proof found independently by P. Deligne [25].

18. K3 surfaces in positive characteristic and their degenerations (1976, 1978, 1981, 1982, 1984). Analytical methods for studying complex algebraic K3 surfaces seemed to be an insurmountable obstacle to the study of these surfaces over fields of positive characteristic. The first breakthrough in this direction was made in M. Artin's paper [7], which, according to I. V. Dolgachev, was viewed by Shafarevich as one of the most brilliant papers he ever read. In this paper, Artin introduced the notion of periods for supersingular K3 surfaces, i.e., for K3 surfaces over an algebraically closed field of positive characteristic all of whose classes in the second étale cohomology are algebraic.

This subject was substantially developed in a large series of papers [105–112] written by Shafarevich jointly with his student A. N. Rudakov, as well as in the study [113] carried out jointly with A. N. Rudakov and T. Zink. In [105], Shafarevich and Rudakov described the relationship between inseparable morphisms and vector fields and used this relationship to prove that, just as in the case of characteristic zero, there are no vector fields on K3 surfaces in positive characteristic (additional related geometric results on elliptic surfaces were obtained in [106, 107]). Note that the absence of vector fields implies the existence of a formal moduli variety for K3 surfaces in positive characteristic.

In [108], Shafarevich and Rudakov proved the unirationality of supersingular K3 surfaces over a field of characteristic 2, constructed a moduli variety for such surfaces with a fixed discriminant of the Néron–Severi lattice, proved the irreducibility of this variety, and found its dimension.

In [111], Shafarevich and Rudakov constructed a moduli variety of supersingular K3 surfaces for an arbitrary characteristic $p > 0$ and described in detail the period map from this moduli variety to the period space constructed previously by A. Ogus [90] in terms of crystalline cohomology. They proved that the period map is étale and, using this property, established that the completeness of the moduli variety is equivalent to the fact that the period map is an isomorphism. In other terms, the latter condition can be expressed as follows: for supersingular K3 surfaces, analogs of Torelli's theorem and the statement on the surjectivity of the period map hold. In this case, the completeness of the moduli variety is equivalent to the fact that any supersingular K3 surface over a generic point of a formal germ of a smooth curve has a smooth model after a finite change of base, i.e., to the fact that supersingular K3 surfaces do not degenerate.

In [111], Shafarevich and Rudakov proved non-degeneracy for supersingular K3 surfaces for $p = 2, 3$ and small values of the discriminant of the Néron–Severi lattice. In [110], they proved non-degeneracy for strongly elliptic supersingular K3 surfaces. Finally, in [113], Shafarevich, Rudakov, and Zink obtained non-degeneracy for arbitrary supersingular K3 surfaces as a consequence of the results on the behavior of the height of the formal Brauer group of a surface under specialization. In [112], this general non-degeneracy was proved by another, more geometric, method. Thus, analogs of Torelli's theorem and statement on the surjectivity of period maps were proved for supersingular K3 surfaces.

19. Families of commutative algebras (1990, 2001). In a later period of his scientific activity, Shafarevich again returned to moduli varieties of algebraic objects (cf. Subsection 13) studying families of finite-dimensional nilpotent commutative algebras (without unity). The presence of continuous parameters in this problem makes it natural to use the methods of algebraic geometry. Namely, the multiplication laws on a given finite-dimensional vector space correspond to the points of some algebraic variety.

In [141], Shafarevich investigated the first nontrivial case: algebras N of dimension n and nilpotency class 2, i.e., with $N^3 = 0$, over an algebraically closed field of characteristic zero. He considered the irreducible components of the variety $\mathcal{A}_n$ of multiplication laws of such algebras and found their dimensions and singular points. He proved that for every positive integer r there exists a unique irreducible component $\mathcal{A}_{n,r}$ of $\mathcal{A}_n$ for which the generic point defines an algebra N with the dimension of the square N^2 equal to r.

Among all irreducible components $\mathcal{A}_{n,r}$, it is natural to distinguish stable ones, i.e., those that are also irreducible components of the variety $\mathcal{C}_n$ of multiplication laws of all commutative associative algebras of the same dimension n. In other words, under small deformations of the algebras corresponding to generic points of stable components, such algebras remain nilpotent of nilpotency class 2.

Let d be the number of generators of the algebra, i.e., the dimension of the space N/N^2, or, which is equivalent, $d = n - r$. Shafarevich proved that the components corresponding to the values $2 < r \le (d-1)(d-2)/6 + 2$ are stable, except possibly the case of $d = 5$ and $r = 4$, while the components with $r > (d^2 - 1)/3$ and $r = 1, 2$ are unstable. Since one always has $r < d(d+1)/2$, the interval of possible values of r is split into three parts that are approximately equal asymptotically in d; the part with smaller values of r corresponds to stable components, the part with greater values, to the unstable components, and the result for the middle part remains unknown.

Every algebra of nilpotency class 2 defines r symmetric $d \times d$ matrices in terms of which the stability criterion is formulated. For $r = 3$, this criterion is closely related to the Barth condition, which is well-known in the theory of vector bundles on the projective plane.

All these constructions and results became one of the first steps in the new theory of deformations and in the classification of finite-dimensional commutative algebras.

Later, in [146], Shafarevich analyzed the degenerations of semisimple algebras. He considered the irreducible component $\mathcal{X}_n$ of $\mathcal{C}_n$ that contains classes of semisimple algebras. He conjectured that $\mathcal{X}_n$ has singularities in codimension 2 and confirmed this conjecture by a number of examples.

20. Finiteness results for K3 surfaces and abelian surfaces (1996). In [143], Shafarevich formulated a conjecture on the finiteness of the set of isomorphism classes of Néron–Severi lattices of complex K3 surfaces defined over a number field of degree at most a given number n. In this paper, he also proved the conjecture for K3 surfaces with the maximum possible rank of the Néron–Severi lattice equal to 20. Moreover, for such K3 surfaces, he proved even a stronger statement saying that the corresponding set of isomorphism classes of the complex surfaces themselves is finite. In addition, he proved a similar statement for abelian surfaces with the maximum possible rank of the Néron–Severi lattice equal to 4, to which the statement on K3 surfaces was reduced.

Note that K3 surfaces with Néron–Severi lattice of rank 20 are a special case of CM-type K3 surfaces defined by Piatetski-Shapiro and Shafarevich [99]. For arbitrary CM-type K3 surfaces, Shafarevich's conjecture on the finiteness of Néron–Severi lattices was proved by M. Orr and A. N. Skorobogatov (a student of Yu. I. Manin) [92]. In addition, recently, together with Yu. G. Zarkhin, they proved [93] that Shafarevich's conjecture for arbitrary K3 surfaces follows from R. Coleman's conjecture on the finiteness of the number of discriminants of endomorphism rings of abelian varieties of fixed dimension defined over number fields of bounded degree.

Following one of his main methods for studying the arithmetic of algebraic varieties, Shafarevich also considered a functional analog of this conjecture. More precisely, the question concerns the finiteness of the set of Néron–Severi lattices of surfaces over the field $\overline{\mathbb{C}(t)}$ from a certain class that are defined over an extension of the field $\mathbb{C}(t)$ of degree at most n. The corresponding statements on K3 surfaces of rank 20 and on abelian surfaces of rank 4 are proved in exactly the same way as their numerical counterparts. In [144, 145], Shafarevich proved the finiteness of the set of isomorphism classes of Néron–Severi lattices for the more complicated case (next in complexity) of K3 surfaces of rank 19 and of abelian surfaces of rank 3. Note that there are two types of rank 3 abelian surfaces: those that are isogenous to the square of an elliptic curve without complex multiplication, and those for which the endomorphism algebra defines a quaternion algebra over $\mathbb{Q}$ that splits over $\mathbb{R}$. In the second case, Shafarevich proved in [144] even the finiteness of the set of such surfaces themselves up to isomorphism, where he used Shimura curves, congruence relations for them, and fine arithmetic properties of orders in quaternion algebras.

21. Tenth discriminant problem (2013). In his last research paper [148], written at the age of 90, Shafarevich revised the ideas of K. Heegner and gave an elementary proof of the tenth discriminant theorem, i.e., of the statement that there exist only nine one-class imaginary quadratic fields. Heegner's proof [45] was mathematically complete but was not presented clearly enough and remained incomprehensible to the mathematical community for more than 15 years. Later, proofs of the tenth discriminant theorem appeared in the papers by H. M. Stark [153] and A. Baker [9], and the original ideas of Heegner were transformed into an indisputable proof by M. Deuring [30] and B. J. Birch [15]. Finally, a completely clear proof, based on more geometric methods, was given by Shafarevich in his course of lectures delivered at MSU in the 1970s.

This proof formed the content of the paper [148], written by the great master with all clarity and elegance of presentation characteristic of him and with the conceptual depth of the approaches used.

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Translated by I. Nikitin