

Elliptic curves over $K = k(t), \mathbb{F}_q(t), \mathbb{C}(t)$
Height moduli, Exact counts, Rank jumps

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Geometry & Topology Seminar

1. Geometry is enlightening and quadratic formula is awesome

$$(x, y) = \left(\frac{t^2 - 1}{t^2 + 1}, \frac{2t}{t^2 + 1} \right) \in \mathbb{Q}^2$$

we parametrized all rational points on the unit circle over $\mathbb{Q}$

2. Integral points on the projective conic

$$C := V(X^2 + Y^2 - Z^2) \subset \mathbb{P}^2$$

are precisely the *Pythagorean triples*:

$$[X : Y : Z] = [a^2 - b^2 : 2ab : a^2 + b^2] \in \mathbb{Z}^3$$

3. Height measures arithmetic complexity: for $\gcd(a, b) = 1$,

$$\text{ht}\left(\frac{a}{b}\right) = \max(|a|, |b|)$$

$$\text{ht}(4/10) = 5 \text{ and}$$

$$\text{ht}(1000000001/1000000000) = 1000000001 \neq 1$$

1. Parametrization needs a starting rational point. For instance, the conic $x^2 + y^2 = 3$ has *no* rational point: $X(\mathbb{Q}) = \emptyset$.
2. Higher degree curves are “Fermat’s Last Theorem”. For $x^4 + y^4 = 1$ (a genus-3 curve), one has by *Wiles-Taylor*

$$X(\mathbb{Q}) = \{(\pm 1, 0), (0, \pm 1)\}$$

More generally, by the *Mordell–Faltings theorem*, if $X/\mathbb{Q}$ is a smooth projective curve of genus ≥ 2 , then $X(\mathbb{Q})$ is finite.

3. For an elliptic curve over $\mathbb{Q}$ in Weierstrass form

$$E : y^2 = x^3 + Ax + B,$$

there is still no general algorithm that, given $A, B \in \mathbb{Q}$, always determines the full set $E(\mathbb{Q})$ explicitly.

Degree of countable infinity, the rank

- (1) (**Mordell–Weil**) For an elliptic curve $E/\mathbb{Q}$, the group of rational points is finitely generated:

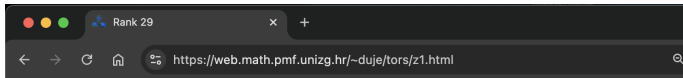
$$E(\mathbb{Q}) \cong \mathbb{Z}^r \oplus E(\mathbb{Q})_{\text{tors}},$$

where $r \geq 0$ is the Mordell–Weil rank and $E(\mathbb{Q})_{\text{tors}}$ is finite.

- (2) Fundamental questions about r :

- ▶ An algorithm that is guaranteed to correctly compute r ?
- ▶ Which values of r can occur? How often do they occur?
- ▶ Is there an upper limit? Can r be arbitrarily large?

- (3) Records: the current best explicit examples with largest rank is Elkies–Klagsbrun (announced 2024) produced an $E/\mathbb{Q}$ with *rank at least 29* (the previous unconditional record was ≥ 28 from Elkies, 2006).



Trivial torsion group, rank ≥ 29

Elkies - Klagsbrun (2024)

$$y^2 + xy = x^3 - 27006183241630922218434652145297453784768054621836357954737385x + 55258058551342376747536699591118191821521067032535079608372404779149413277716173425636721497$$

Independent points of infinite order:

$P_1 = [2891195474228537189458255536634, 1159930748096124706459835910727318679593425283]$
 $P_2 = [3402542165322127811451484642234, 1661508223164691055862657623730465560755290883]$
 $P_3 = [4298760026558467240422107564794, 4313142249890236204790986787384907722927474563]$
 $P_4 = [3728756667770947009884455714554, 2530180219584734091116528693531660545660397443]$
 $P_5 = [5991744132052078230511185130234, 10418901628842034362301273055728300669218858883]$
 $P_6 = [3236493534632768520540227223034, 1324626796262167243658687198416201825373745283]$
 $P_7 = [78226686134991174232380689386234, 690394210062759896503429654125516779999512554883]$
 $P_8 = [11492605643548859374635605140234, 35536316911450952155461624238308456029618940883]$
 $P_9 = [-5143303362384229804906088118566, 762235651110798686412035235567430568022368483]$
 $P_{10} = [443985655575065435281568435002, 6584468124388858623214803939643557365635620355]$
 $P_{11} = [-979565018904269680752629749766, 8987348422104537684966706438714038633832170883]$
 $P_{12} = [5184894285212178249566461261834, 7390536788003150201273204464695859875505480483]$
 $P_{13} = [-4469171023687146502067179612166, 9310658892841458934133221137392081403414455683]$
 $P_{14} = [3606405835110925482450522970234, 218364466698170363248266219339048004012789883]$
 $P_{15} = [16151744576785317732688993162234, 61908882092472338946519909276455831463747210883]$
 $P_{16} = [3573684355943766387962362869754, 2094467155115749424853047283659077805560259203]$
 $P_{17} = [-759376049938858166436491644166, 8679171135458197195914024161800061810952119683]$
 $P_{18} = [-5328058719935886182106003119366, 6920588147379497633202935557367499676224350083]$
 $P_{19} = [5380268474895377355583039694554, 8105660240030025092450118297303424395856037443]$
 $P_{20} = [17069233487425098088940203248484, 67583677272795299213867443505411893525786510633]$
 $P_{21} = [5215432542403430758248050783794, 7501515746204716855921710958364078294243814643]$
 $P_{22} = [2838942178004620439763692432122, 1212346280964590308944175800544505700108208003]$
 $P_{23} = [243146882395382015946366404808154/81, 811625272160726332199288136187427505366582108107/729]$
 $P_{24} = [2558229016839511149831260080762, 17065983958300799943875052441333827096496371203]$
 $P_{25} = [261253942905600010977556672634, 2157503396243552448798851089310708763298766083]$
 $P_{26} = [23678312077644931683114439906234, 1462722361020796436741527433473386115047618883]$
 $P_{27} = [3379397084927230910084852603902, 1608494167359575995485655188349208450365853755]$
 $P_{28} = [363248773087098917912491355514, 2255654937037700801978158381185619053396712963]$
 $P_{29} = [2428778263277521959543043930234, 1998325023610603606161737305486867803334410883]$

Previous record with [rank \$\geq 28\$](#)

Fix a height $ht(E)$ e.g. naive height for a minimal Weierstrass model $E : y^2 = x^3 + Ax + B$ with no $p^4|A$ and $p^6|B$:

$$ht(E) := \max\{4|A|^3, 27|B|^2\}$$

$$\mathcal{E}(\leq X) := \{E/\mathbb{Q} \text{ up to } \mathbb{Q}\text{-isomorphism} : ht(E) \leq X\}$$

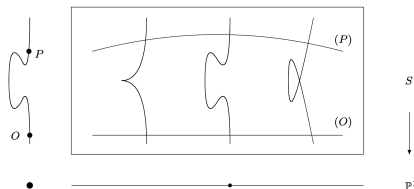
Goldfeld–Katz–Sarnak philosophy

As $X \rightarrow \infty$, ranks 0 and 1 should each account for one half of the family, while curves of rank at least 2 should have density 0.

To average rank, first count $\mathcal{N}(\mathbb{Q}, X) = |\mathcal{E}(\leq X)|$

Elliptic surfaces as families of elliptic curves

The study of **fibrations of curves** lies at the heart of the Enriques–Kodaira classification of algebraic surfaces.



Let k be a perfect field, $K = k(t)$, and $C = \mathbf{P}_k^1$. An elliptic curve E/K extends, uniquely up to isomorphism, to a relatively-minimal elliptic surface

$$f : X \longrightarrow C$$

with zero section $O : C \rightarrow X$, where X is a smooth projective surface, f is proper and flat, and

$$(X_\eta, O_\eta) \cong (E, O_E).$$

Relative-minimality means that no fibre contains a (-1) -curve.

God Bless Universal Family!

Fix a field k . A **family of stable elliptic curves with a section** over $\mathbb{P}_k^1$ is equivalent to a morphism $\varphi : \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}$.

$$\begin{array}{ccccc} X & \xrightarrow{\nu} & Y := \varphi^*(\overline{\mathcal{E}}_{1,1}) & \longrightarrow & \overline{\mathcal{E}}_{1,1} \\ \downarrow f & & \downarrow g & & \downarrow p \\ \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 & \xrightarrow{\varphi} & \overline{\mathcal{M}}_{1,1} \end{array}$$

Here $p : \overline{\mathcal{E}}_{1,1} \rightarrow \overline{\mathcal{M}}_{1,1}$ is the universal stable elliptic curve, $g : Y \rightarrow \mathbb{P}^1$ is the pulled-back stable family, and $\nu : X \rightarrow Y$ is the relatively-minimal resolution (A_k singularity to I_{k+1} fiber).

Equivalently, writing $\overline{\mathcal{M}}_{1,1} \simeq \mathcal{P}(4, 6)$, a morphism φ is given by

$$\varphi = (a_4(u, v), a_6(u, v)), \quad \deg(a_4) = 4n, \deg(a_6) = 6n,$$

with $\text{Res}(a_4, a_6) \neq 0$ (so a_4, a_6 do not vanish simultaneously).

Faltings height controls the surface

Let $\mathcal{L} = f_*\omega_{X/\mathbb{P}^1}$ be the fundamental line bundle. Then

$$n = \text{ht}(E) = \deg(\mathcal{L}) = \chi(\mathcal{O}_X), \quad \deg \Delta_{\min}(E) = 12n.$$

height	surface	dimension of the moduli
$n = 1$	rational elliptic surface	8
$n = 2$	elliptic K3 surface	18
$n \geq 3$	properly elliptic surface	$10n - 2$

1. $n = 1$ (**rational elliptic surface**). Then X is rational and $\kappa(X) = -\infty$. Generically, Δ has 12 simple zeros, i.e. 12 nodal fibres (type I_1). One construction: take a pencil of plane cubics in $\mathbb{P}^2$ and blow up its 9 base points.
2. $n = 2$ (**elliptic K3 surface**). Then X is a projective K3 surface with an elliptic fibration, hence $\kappa(X) = 0$. Generically, Δ has 24 simple zeros, i.e. 24 nodal fibres; moduli $\dim = 18$.

Isotrivial rational elliptic surface of height $n = 1$

Isotrivial Rational Elliptic Surface $n = d + \sum_{i=1}^s a_{i/2}$

$$n = 7 = 1/6 + 5/6$$

$$\begin{cases} a_4 = 0 \\ a_6 = u \cdot v^5 \end{cases}$$

$$\begin{cases} d = 0 \\ a_{1/6} = 1/6, a_{5/6} = 5/6 \end{cases}$$

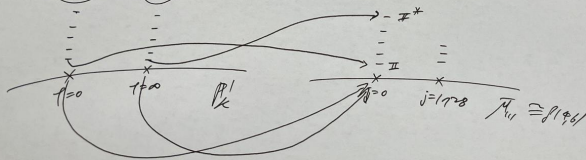
$$v/a_4 = \infty \text{ minimal}$$

$$[u:v] \quad u_0 = 1$$

$$v/a_6 = \begin{cases} 7 < 6 \text{ at } u=0 \Rightarrow v=1 & [0:1] \Leftrightarrow t=0 \\ 5 < 6 \text{ at } v=0 \Rightarrow u=1 & [1:0] \Leftrightarrow t=\infty \end{cases}$$

$$\Delta = 27 u^2 v^{10} - \text{deg } 12$$

$$j \equiv 0$$



$$y^2 = x^3 + u v^5 \in \mathbb{S}(1/6) \cdot \frac{7}{6} \cdot \frac{5}{6} \cdot \frac{1}{2}$$

$$\downarrow u = z^6$$

$$v = q^{12}$$

$$y^2 = x^3 + (z^6) v^5$$

$$\because \chi_{11} = 7$$

$$\downarrow v = w^6$$

$$y^2 = x^3 + (w^6) v^5$$

$$\because \chi_{11} = 5$$

$$y^2 = x^3 + 7$$

Twisted sources restore the missing fibre data

The regular coarse map $j : \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1} \cong \mathbb{P}^1$ forgets

$$\gamma_x = (\nu_x(a_4), \nu_x(a_6))$$

The classifying map extends uniquely after making the source stacky:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\varphi} & \overline{\mathcal{M}}_{1,1} \\ \pi \downarrow & \nearrow & \\ \mathbb{P}^1 & & \end{array}$$

At its stacky points, record $\Gamma = ((r_1, a_1), \dots, (r_s, a_s))$, where μ_{r_i} acts on the Hodge line by the character a_i .

The stabilizers restore the fibre data discarded by coarse j .

Tate's algorithm via twisted morphisms

Assume $\text{char}(k) \neq 2, 3$ and put

$$m = \min\{3\nu(a_4), 2\nu(a_6)\}$$

Then the local twisting data is

$$(r, a) = \left(\frac{12}{\gcd(m, 12)}, \frac{m}{\gcd(m, 12)} \right).$$

$$(\nu(a_4), \nu(a_6)) = (1, \geq 2) \iff (r, a) = (4, 1) \iff \text{type III}$$

$$\boxed{\text{vanishing data}} \iff \boxed{\text{twisting data}} \iff \boxed{\text{singular fibre}}$$

Stacky height is a fine moduli data carrying its fibre invariants.

Vanishing $\gamma \Leftrightarrow$ Reduction type $\Theta \Leftrightarrow$ Twisting Γ

Theorem (Bejleri–Park–Satriano; April 2024)

If $\text{char}(K) \neq 2, 3$. Then the twisting condition (r, a) and the order of vanishing of j at $j = \infty$ determine the Kodaira fiber type, and (r, a) is in turn determined by $m = \min\{3\nu(a_4), 2\nu(a_6)\}$.

$\gamma : (\nu(a_4), \nu(a_6))$	Reduction type with $j \in \overline{M}_{1,1}$	$\Gamma : (r, a)$
$(\geq 1, 1)$	II with $j = 0$	$(6, 1)$
$(1, \geq 2)$	III with $j = 1728$	$(4, 1)$
$(\geq 2, 2)$	IV with $j = 0$	$(3, 1)$
$(2, 3)$	$I_{k>0}^*$ with $j = \infty$ I_0^* with $j \neq 0, 1728$	$(2, 1)$
$(\geq 3, 3)$	I_0^* with $j = 0$	$(2, 1)$
$(2, \geq 4)$	I_0^* with $j = 1728$	$(2, 1)$
$(\geq 3, 4)$	IV^* with $j = 0$	$(3, 2)$
$(3, \geq 5)$	III^* with $j = 1728$	$(4, 3)$
$(\geq 4, 5)$	II^* with $j = 0$	$(6, 5)$

Stacky height splits into global and local pieces

For the twisted extension (root stack) $\varphi : \mathcal{C} \rightarrow \mathcal{X}$,

$$\mathrm{ht}_{\mathcal{L}}(P) = \underbrace{d}_{\text{stable height}} + \underbrace{\sum_{i=1}^s \frac{a_i}{r_i}}_{\text{local contributions}}$$

Height moduli theorem - Bejleri-Park-Satriano

For fixed n , there is a separated finite-type Deligne–Mumford stack

$$\mathcal{M}_{n,C}(\mathcal{X}, \mathcal{L})$$

whose k -points are the $K = k(C)$ points of stacky height n . It is stratified by the possible twisting data Γ .

Are Mordell–Weil ranks over $\mathbb{C}(t)$ unbounded?

For a Jacobian elliptic surface of height $n \geq 1$, Shioda–Tate and the Lefschetz bound give

$$\mathrm{rk} E_S(\mathbb{C}(t)) \leq \rho(S) - 2 \leq 10n - 2.$$

It remains open whether there is a bound independent of n .

Classical high-rank examples

construction	height	rank	singular fibres
Shioda: $y^2 = x^3 + t^{360} + 1$	60	68	type II, isotrivial with $j = 0$
Shioda–Stiller: $y^2 = x^3 + x + t^{1260}$	210	56	2520 fibres of type I_1 , non-isotrivial

$\mathbb{Q}$	$\mathbb{C}(t)$	$\mathbb{F}_q(t)$
open	open	unbounded for every fixed q

Tate–Shafarevich: isotrivial examples in odd characteristic. Ulmer: non-isotrivial examples in every positive characteristic.

Precise proportions of E/K motivated by NT

Theorem (Generic Torsion Freeness; Phillips)

The set of torsion-free elliptic curves over global function fields has density 1. i.e., 'Most elliptic curves over K are torsion free'.

Theorem (Boundedness; Tate-Shafarevich & Ulmer)

*The ranks of non-constant elliptic curves over $\mathbb{F}_q(t)$ are unbounded (in both the **isotrivial** and the **non-isotrivial** cases).*

Ulmer's non-isotrivial elliptic curve of infinite rank

1. Start with $y^2 + xy = x^3 - t^d$, then *complete the square* via $y = y' - \frac{x}{2}$ and then *complete the cubic* via $x = x' - \frac{1}{12}$. We need $\text{char}(k) \neq 2, 3$ to get to the short Weierstrass form.
2. We get $y^2 = x^3 - \frac{1}{48}x + \frac{1}{864} - t^d$. Coefficients should be integral thus we take $\lambda = 2 \cdot 3$ to multiply λ^4 to $-\frac{1}{48}$ and λ^6 to $+\frac{1}{864} - t^d$.
3. We arrive at $y^2 = x^3 - 27x + 54 - 2^6 \cdot 3^6 \cdot t^d$ thus $[-\frac{1}{48} : \frac{1}{864} - t^d] = [-27 : 54 - 2^6 \cdot 3^6 \cdot t^d]$.
4. Remember the isomorphism, for any $\lambda \in \mathbb{G}_m$

$$[y^2 = x^3 + Ax + B] \cong [y^2 = x^3 + \lambda^4 \cdot Ax + \lambda^6 \cdot B]$$

via $x \mapsto \lambda^{-2} \cdot x$ and $y \mapsto \lambda^{-3} \cdot y$ by the *Weighted homogeneous coordinate* of $\mathcal{P}(4, 6)$.

Enter your code in the box below. Click on "Submit" to have it evaluated by Magma.

```
KK<t> := FunctionField(GF(4007));
E := EllipticCurve([-27, 54 - 2^6*3^6*t^11]);
E;
&*BadPlaces(E);
LocalInformation(E);
```

Cancel

Submit

```
Elliptic Curve defined by  $y^2 = x^3 + 3980x + (1428t^{11} + 54)$  over Univariate
rational function field over GF(4007)
t^11 + 1549
[ <(t^5 + 3335*t^4 + 2186*t^3 + 488*t^2 + 2393*t + 906), 1, 1, 1, II, false>,
  <(t^5 + 3337*t^4 + 2186*t^3 + 488*t^2 + 3369*t + 906), 1, 1, 1, II, false>,
  <(t), 11, 1, 11, II, true>, <(1/t), 2, 2, 1, II, true>, <(t + 1342), 1, 1, 1,
  II, false> ]
```

1. The corresponding elliptic surface has a fiber of Kodaira type I_d at zero (at $t = 0$), while the fiber at infinity (at $1/t = 0$) is given by the congruence class $\bar{d}$ of d modulo 6 : $(\bar{d}, \Theta)$
 $(\bar{0}, I_0)$ $(\bar{1}, II^*)$ $(\bar{2}, IV^*)$ $(\bar{3}, I_0^*)$ $(\bar{4}, IV)$ $(\bar{5}, II)$
2. Outside char 2, 3, there are d fibres of type I_1 at the zeroes of $432t^d - 1$ (some of which may be merged if $\text{char}(k)|d$).

The aim of this paper is to produce elliptic curves over $K = \mathbb{F}_p(t)$ which are nonisotrivial ($j \notin \mathbb{F}_p$) and which have arbitrarily large rank.

THEOREM 1.5. *Let p be an arbitrary prime number, $\mathbb{F}_p$ the field of p elements, and $\mathbb{F}_p(t)$ the rational function field in one variable over $\mathbb{F}_p$. Let E be the elliptic curve defined over $K = \mathbb{F}_p(t)$ by the Weierstrass equation*

$$y^2 + xy = x^3 - t^d$$

where $d = p^n + 1$ and n is a positive integer. Then $j(E) \notin \mathbb{F}_p$, the conjecture of Birch and Swinnerton-Dyer holds for E over K , and the rank of $E(K)$ is at least $(p^n - 1)/2n$.

Ulmer's proof shows that as the height of Ulmer's curve goes up as $n = 1 + \lfloor \frac{d-1}{6} \rfloor \rightarrow \infty$, the algebraic/analytic rank r goes up to ∞ .

Sketch of Ulmer's proof

1. Construct an elliptic surface $S \rightarrow \mathbb{P}^1$ over $\mathbb{F}_p$ with generic fiber $E : y^2 + xy = x^3 - t^d$ for $d = p^n + 1$ and $n \in \mathbb{Z}_+$.
2. Construct (and carefully study) a birational isomorphism between S and F_d/G , the quotient of a Fermat surface i.e. $V(x^d + y^d + z^d + w^d) \subset \mathbb{P}^3$ ($d = 4$ then it is K3 surface).
3. Using the fact that the Tate conjecture for surfaces is known for Fermat surfaces, one can deduce the Tate conjecture for S .
4. Use the fact that the Tate conjecture for S implies the Birch and Swinnerton-Dyer conjecture for E . Thus the ranks of the elliptic curves in the family all equal their analytic ranks.
5. The analytic ranks can be computed by relating the L-function of E to the zeta function of S , which can be related to the zeta function of F_d , which is known by Gauss sum computation of Weil. From this one is able to compute the analytic rank which is unbounded from below.

Shioda–Tate and Mordell–Weil rank

For a Jacobian elliptic surface $S \rightarrow \mathbb{P}_{\mathbb{C}}^1$, the Shioda–Tate formula gives

$$\rho(S) = \underbrace{2}_{\text{fibre and zero section}} + \underbrace{\sum_v (m_v - 1)}_{\text{reducible fibres}} + \text{rk } E_S(\mathbb{C}(t)).$$

Here $\rho(S) = \text{rk NS}(S)$ and m_v is the number of irreducible components of the fibre over v .

On the squarefree-discriminant locus

Every singular fibre has type I_1 . Thus $m_v = 1$ for every v , and

$$\rho(S) = 2 + \text{rk } E_S(\mathbb{C}(t)).$$

Consequently, while every singular fibre remains of type I_1 ,

Picard-rank jump $\iff$ Mordell–Weil rank jump.

Integral $(1, 1)$ -classes independent modulo the fibre and zero-section classes give independent Mordell–Weil sections.

Ramification points and period equations

Let $n \geq 2$. Suppose every singular fibre of S has type I_1 and the classifying map

$$\varphi_S : \mathbb{P}^1 \longrightarrow \overline{\mathcal{M}}_{1,1}$$

has reduced stack-theoretic ramification divisor. Its support consists of $N = 10n - 2$ distinct points $p_1, \dots, p_N$. Each p_i determines a deformation direction t_i and a class η_i in primitive $(1, 1)$ -cohomology.

Period equations

Let γ be an integral class orthogonal to the fibre and zero-section classes, and choose a local holomorphic frame $\omega_0(t), \dots, \omega_{n-2}(t)$ of holomorphic two-forms. Put

$$\Phi_\gamma(t) = (\gamma \cdot \omega_0(t), \dots, \gamma \cdot \omega_{n-2}(t)).$$

The Lefschetz $(1, 1)$ theorem gives

$$\gamma \text{ is algebraic on } S_t \iff \Phi_\gamma(t) = 0.$$

One integral class gives $n - 1$ period equations. Evaluation at distinct ramification points gives the Vandermonde minors proving transversality.

Nearby elliptic surfaces acquire independent sections

Analytically dense Mordell-Weil rank jumps - Park

Let $n \geq 2$ and let r be an integer satisfying

$$1 \leq r \leq \left\lfloor \frac{10n-2}{n-1} \right\rfloor$$

Every analytic neighbourhood of every minimal Jacobian elliptic surface of height n contains infinitely many pairwise non-isomorphic surfaces with only I_1 singular fibres and r independent sections.

Canonical heights of the chosen sections can be made arbitrarily large.

The transverse range is determined by

$$\underbrace{r(n-1)}_{\text{period equations}} \leq \underbrace{10n-2}_{\text{deformation directions}}$$

Thus the theorem applies to

$$r \leq 18 \quad (n=2), \quad r \leq 14 \quad (n=3), \quad r \leq 10 \quad (n \geq 10)$$

Thank you!