

Arithmetic of the Moduli of Fibrations

Arithmetic Moduli of Elliptic Surfaces

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While this is the most general setup, it is natural to work with the case when the base curve is the smooth projective line \mathbb{P}^1 and there exists a section $S : \mathbb{P}^1 \hookrightarrow X$ coming from the identity points of the elliptic fibres and not passing through the singular points.

Elliptic surfaces over \mathbb{P}^1 with a section

Here we list the properties of an elliptic surface X with discriminant degree $12n$. This also works for any field K with $\text{char}(K) \neq 2, 3$.

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1. When $n = 1$, X is a **Rational elliptic surface** with the Kodaira dimension $\kappa = -\infty$ which has 12 nodal singular fibers generically. It is acquired from a pencil of cubic curves in \mathbb{P}^2 by blowing up a base locus of nine points coming from the intersection of two general cubic curves.

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2. When $n = 2$, X is a $K3$ surface with an elliptic fibration (i.e., **Elliptic K3 surface**) which has the Kodaira dimension $\kappa = 0$ that has 24 nodal singular fibers generically. Note that X is a minimal surface.

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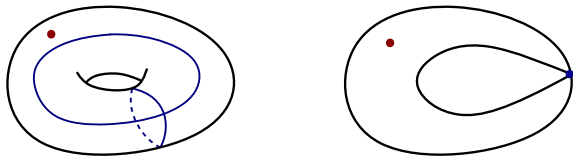
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3. When $n \geq 3$, X is called a **Properly elliptic surface** with Kodaira dimension $\kappa = 1$ that has $12n$ nodal singular fibers generically. Note that X is also a minimal surface.

Deligne–Mumford stack $\overline{\mathcal{M}}_{1,1}$ of stable elliptic curves

Let us recall that $\overline{\mathcal{M}}_{1,1}$ is a smooth proper Deligne–Mumford stack of stable elliptic curves with a coarse moduli space $\overline{M}_{1,1} \cong \mathbb{P}^1$. This \mathbb{P}^1 parametrizes the j -invariants of elliptic curves.

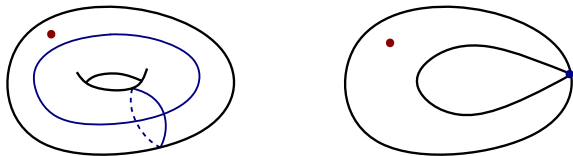
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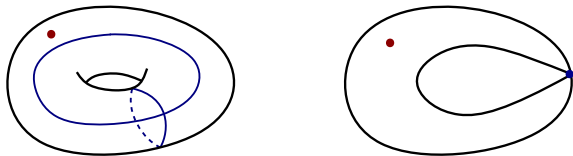
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When the characteristic of the field K is not equal to 2 or 3, $(\overline{\mathcal{M}}_{1,1})_K \cong [(Spec K[a_4, a_6] - (0, 0))/\mathbb{G}_m] =: \mathcal{P}_K(4, 6)$ through the short Weierstrass equation: $Y^2 = X^3 + a_4X + a_6$

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Stabilizers are the orbifold points $[1 : 0]$ & $[0 : 1]$ with μ_4 & μ_6 respectively and the generic stacky points such as $[1 : 1]$ with μ_2

Moduli stack of stable elliptic surfaces

The fine moduli $\overline{\mathcal{M}}_{1,1}$ comes with universal family $p : \overline{\mathcal{C}}_{1,1} \rightarrow \overline{\mathcal{M}}_{1,1}$ of stable elliptic curves. Thus, a stable elliptic surface $g : Y \rightarrow \mathbb{P}^1$ is induced from a morphism $\varphi_f : \mathbb{P}^1 \rightarrow \overline{\mathcal{M}}_{1,1}$ and vice versa.

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$$\begin{array}{ccccc} X & \xrightarrow{\nu} & Y = \varphi_f^*(\overline{\mathcal{C}}_{1,1}) & \longrightarrow & \overline{\mathcal{C}}_{1,1} \\ \downarrow f & & \downarrow g & & \downarrow p \\ \mathbb{P}^1 & \xlongequal{\quad} & \mathbb{P}^1 & \xrightarrow{\varphi_f} & \overline{\mathcal{M}}_{1,1} \end{array} \quad (1)$$

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Proposition (Changho Han, J.-Y. Park)

The moduli stack of stable elliptic surfaces over \mathbb{P}^1 with $12n$ nodal singular fibers and a section is the Hom stack $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$.

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Here, we fixed the parameterization of the domain \mathbb{P}^1 which is good for 'Global Fields Analogy' (since \mathbb{Q} has the *unique* ring of integers called \mathbb{Z}) but not natural from Geometric perspective.

Group actions on stacks for stack quotients of stacks

It is natural to consider the action of $\text{Aut}(\mathbb{P}^1) = \text{PGL}_2$ on $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ by composing the stable elliptic surface $g : Y \rightarrow \mathbb{P}^1$ with an automorphism of \mathbb{P}^1 .

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It is easy to see that this action is induced by an action on the ambient weighted projective stack $\mathcal{P}(V)$.

$$(A, B) \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4n)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(6n)) =: V \quad (2)$$

define the so-called *Weierstrass data* of the fibration. Indeed, the action of an element of PGL_2 on the homogeneous coordinates X, Y of \mathbb{P}^1 translates to an action on the global sections A, B of $\mathcal{O}_{\mathbb{P}^1}(4n), \mathcal{O}_{\mathbb{P}^1}(6n)$ which are the homogeneous coordinates of $\mathcal{P}(V) = \mathcal{P}(\underbrace{4, \dots, 4}_{4n+1 \text{ times}}, \underbrace{6, \dots, 6}_{6n+1 \text{ times}}) \in \mathbb{Z}^{10n+2}$.

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Note that since both $\text{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ and $\mathcal{P}(V)$ are themselves stacks, the formal definition of these actions requires one to use the notion of group actions on stacks presented in [Romagny].

Group actions on stacks for stack quotients of stacks

We have the following commutative diagram

$$\begin{array}{ccccc} \mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}) & \hookrightarrow & \mathrm{Rat}_n^\gamma(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1}) & \hookrightarrow & \mathcal{P}(V) \\ \downarrow / \mathrm{PGL}_2 & & \downarrow / \mathrm{PGL}_2 & & \downarrow / \mathrm{PGL}_2 \\ \mathcal{W}_{\mathrm{sf},n} & \hookrightarrow & \mathcal{W}_{\mathrm{min},n} & \hookrightarrow & [\mathcal{P}(V) / \mathrm{PGL}_2] \end{array}$$

where the horizontal arrows are open embeddings of moduli stacks.

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Theorem (Johannes Schmitt, J.-Y. Park)

Fix a degree $n \in \mathbb{Z}_{\geq 1}$ and a base field K with $\mathrm{char}(K) \neq 2, 3$. Inside the quotient stack $[\mathcal{P}(V) / \mathrm{PGL}_2]$, the open substacks $\mathcal{W}_{\mathrm{min},n}$ (for $n \geq 2$) of minimal Weierstrass fibrations and $\mathcal{W}_{\mathrm{sf},n}$ (for $n \geq 1$) of stable Weierstrass fibrations are smooth, irreducible and separated Deligne–Mumford stacks of finite type with affine diagonal for $\mathrm{char}(K) \nmid n$, which are tame for $\mathrm{char}(K) > 12n$.

Group actions on stacks for stack quotients of stacks

The Weierstrass fibration associated to $[A : B] = [X^{4n} : Y^{6n}]$ is invariant under scaling X by an element of μ_{4n} and by scaling Y under an element of μ_{6n} . Together, these transformations generate a copy of μ_{12n} inside PGL_2 which acts as an automorphism of the fibration, and the quotient is not tame when $\mathrm{char}(K)$ divides $12n$.

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The stack $\mathcal{W}_{\min,1}$ contains points with positive dimensional stabilizers, thus it is no longer of Deligne–Mumford type. These points are precisely PGL_2 -orbit of the Weierstrass data $[A : B]$

$$[A : B] = [0 : XY^5], [XY^3 : 0], [0 : X^2Y^4] \text{ and } [a_0X^2Y^2 : a_1X^3Y^3],$$

where in each case we have a nontrivial action of \mathbb{G}_m on the coordinates X, Y fixing the fibrations. They are the four types of rational elliptic surfaces with two singular fibres

$[\mathrm{II}, \mathrm{II}^*], [\mathrm{III}, \mathrm{III}^*], [\mathrm{IV}, \mathrm{IV}^*], [\mathrm{I}_0^*, \mathrm{I}_0^*]$ both of which are additive type in dual pair. One can see that the open substack $\mathcal{W}'_{\min,1}$ of $\mathcal{W}_{\min,1}$ obtained by removing these four points is indeed Deligne–Mumford for $\mathrm{char}(K) \nmid n$ and tame for $\mathrm{char}(K) > 12$.

Grothendieck ring $K_0(\text{Stck}_K)$ of K -stacks

Ekedahl in 2009 introduced the Grothendieck ring $K_0(\text{Stck}_K)$ of algebraic stacks extending the classical Grothendieck ring $K_0(\text{Var}_K)$ of varieties first defined by Grothendieck in 1964.

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Definition

Fix a field K . Then the *Grothendieck ring $K_0(\text{Stck}_K)$ of algebraic stacks of finite type over K all of whose stabilizer group schemes are affine* is an abelian group generated by isomorphism classes of algebraic stacks $\{\mathcal{X}\}$ modulo relations:

- ▶ $\{\mathcal{X}\} = \{\mathcal{Z}\} + \{\mathcal{X} \setminus \mathcal{Z}\}$ for $\mathcal{Z} \subset \mathcal{X}$ a closed substack,
- ▶ $\{\mathcal{E}\} = \{\mathcal{X} \times \mathbb{A}^n\}$ for \mathcal{E} a vector bundle of rank n on \mathcal{X} .

Multiplication on $K_0(\text{Stck}_K)$ is induced by $\{\mathcal{X}\}\{\mathcal{Y}\} := \{\mathcal{X} \times_K \mathcal{Y}\}$.

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The weighted point count of \mathcal{X} over \mathbb{F}_q is defined as a sum:

$\#_q(\mathcal{X}) := \sum_{x \in \mathcal{X}(\mathbb{F}_q)/\sim} \frac{1}{|\text{Aut}(x)|}$ where $\mathcal{X}(\mathbb{F}_q)/\sim$ is the set of \mathbb{F}_q -isomorphism classes of \mathbb{F}_q -points of \mathcal{X} .

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When $K = \mathbb{F}_q$, the point counting measure $\{\mathcal{X}\} \mapsto \#_q(\mathcal{X})$ gives a well-defined ring homomorphism $\#_q : K_0(\text{Stck}_{\mathbb{F}_q}) \rightarrow \mathbb{Q}$.

Motive/Point count of $\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ over finite fields

Theorem (Changho Han, Hunter Spink, Johannes Schmitt, J.)

The class $\{\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\}$ in $K_0(\mathrm{Stck}_K)$ for $\mathrm{char}(K) \neq 2, 3$ of the moduli stack for stable elliptic fibrations over the parameterized \mathbb{P}^1 with $12n$ nodal singular fibers and a section is equivalent to

$$\{\mathrm{Hom}_n(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\} = \mathbb{L}^{10n+1} - \mathbb{L}^{10n-1}$$

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and over an unparameterized \mathbb{P}^1 with an odd degree n

$$\{\mathcal{W}_{\mathrm{sf},n}\} = \mathbb{L}^{10n-2}$$

where $\mathbb{L} = \{\mathbb{A}^1\}$ is the Lefschetz motive and $\{\mathrm{PGL}_2\} = \mathbb{L}(\mathbb{L}^2 - 1)$.

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Then, by using $\#_q : K_0(\mathrm{Stck}_{\mathbb{F}_q}) \rightarrow \mathbb{Q}$ to count \mathbb{F}_q -points when $\mathrm{char}(\mathbb{F}_q) \neq 2, 3$, we acquire the weighted point counts of the moduli of stable elliptic surfaces over (un)parameterized \mathbb{P}^1 .

Motive/Point count of $\text{Rat}_n^\gamma(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})$ over the finite fields

Theorem (work in progress; Dori Bejleri, Matthew Satriano, J.)

Let $\text{char}(K) \neq 2, 3$. Then the motive for the moduli stack of minimal elliptic surfaces over the parameterized \mathbb{P}^1 with a section and discriminant degree $12n$ having one additive singular fiber is

Fiber type, j	$\{\text{Rat}_n^\gamma(\mathbb{P}^1, \overline{\mathcal{M}}_{1,1})\}/\{\text{PGL}_2\}$
II, $j = 0$	$\mathbb{L}^{10n-3} - \mathbb{L}^{4n-3}$
III, $j = 1728$	$\mathbb{L}^{10n-4} - \mathbb{L}^{6n-4}$
IV, $j = 0$	$\mathbb{L}^{10n-5} - \mathbb{L}^{4n-3}$
$I_{k \geq 0}^*$, $j = \infty$ I_0^* , $j \neq 0, 1728$	$\mathbb{L}^{10n-6} - \mathbb{L}^{10n-7} + \mathbb{L}^{6n-3} - \mathbb{L}^{6n-4} - \mathbb{L}^{4n-3} + \mathbb{L}^{4n-2}$
I_0^* , $j = 1728$	$\mathbb{L}^{10n-7} - \mathbb{L}^{6n-3}$
I_0^* , $j = 0$	$\mathbb{L}^{10n-7} - \mathbb{L}^{4n-4}$
IV^* , $j = 0$	$\mathbb{L}^{10n-8} - \mathbb{L}^{6n-4}$
III^* , $j = 1728$	$\mathbb{L}^{10n-9} - \mathbb{L}^{4n-3}$
II^* , $j = 0$	$\mathbb{L}^{10n-10} - \mathbb{L}^{6n-4}$

Counting elliptic curves over global fields

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Definition

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As the discriminant divisor $\Delta(X)$ of degree $12n$ is an invariant of the choice of semistable model $f : X \rightarrow \mathbb{P}^1$, we count the number of isomorphism classes of semistable elliptic surfaces on the function field $\mathbb{F}_q(t)$ by the bounded height of $\Delta(X)$.

$$ht(\Delta(X)) = \prod_{i=1}^{\mu} |\mathbb{F}_q|^{k_i} = q^{k_1} \dots q^{k_i} \dots q^{k_\mu} = q^{k_1 + \dots + k_\mu} = q^{12n}$$

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Now consider $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) =$

$$|\{\text{Semistable elliptic curves over } \mathbb{P}_{\mathbb{F}_q}^1 \text{ with } 0 < ht(\Delta) \leq \mathcal{B}\}|.$$

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Theorem (Changho Han, J.-Y. Park)

The counting $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B})$ by $ht(\Delta) = q^{12n} \leq \mathcal{B}$ satisfies

$$\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) \leq 2 \cdot \frac{(q^{11} - q^9)}{(q^{10} - 1)} \cdot (\mathcal{B}^{\frac{5}{6}} - 1)$$

which is an equality when $\mathcal{B} = q^{12n}$ for some $n \in \mathbb{N}$ implying that the acquired upper bound is a sharp enumeration, i.e., the upper bound is equal to the function at infinitely many values of $\mathcal{B} \in \mathbb{N}$.

Counting elliptic curves over global fields

Now consider $\mathcal{Z}_{\mathbb{F}_q(t)}(\mathcal{B}) =$

$$|\{\text{Semistable elliptic curves over } \mathbb{P}_{\mathbb{F}_q}^1 \text{ with } 0 < ht(\Delta) \leq \mathcal{B}\}|.$$

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Note that we have the lower order term of zeroth order (constant).

Counting elliptic curves over global fields

Theorem (work in progress; Dori Bejleri, Matthew Satriano, J.)

If $\text{char}(\mathbb{F}_q) > 3$, then the function $\mathcal{Z}_{\mathbb{F}_q(t)}^\gamma(\mathcal{B})$, which counts the number of minimal elliptic curves with one additive singular fiber of γ type over the parameterized $\mathbb{P}_{\mathbb{F}_q}^1$ ordered by $0 < ht(\Delta) = q^{12n} \leq \mathcal{B}$, satisfies:

$$\mathcal{Z}_{\mathbb{F}_q(t)}^\gamma(\mathcal{B}) \leq a_q \cdot \mathcal{B}^{\frac{5}{6}} + b_q \cdot \mathcal{B}^{\frac{1}{3}} + c_q, \text{ when } \gamma = \text{II}, \text{II}^*, \text{IV}, \text{IV}^* \text{ or } \text{I}_0^*$$

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which is an equality when $\mathcal{B} = q^{12n}$ with $n \in \mathbb{Z}_{\geq 1}$ implying that the upper bounds are sharp enumerations.

Heuristics for counting elliptic curves over number fields

Switching to the number field realm with $K = \mathbb{Q}$ and $\mathcal{O}_K = \mathbb{Z}$, one could choose the minimal integral Weierstrass model with the given discriminant divisor Δ which is already a number.

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In order to match the counting with the function field, we define the $ht(\Delta)$ to be the cardinality of ring of functions on subscheme $\text{Spec}(\mathbb{Z}/(\Delta))$. This leads to the following analogue $\mathcal{Z}_{\mathbb{Q}}(\mathcal{B})$.

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Conjecture (work in progress; Heuristic on $\mathcal{Z}_{\mathbb{Q}}(\mathcal{B})$)

The function $\mathcal{Z}_{\mathbb{Q}}(\mathcal{B})$, which counts the number of elliptic curves with at least one additive bad reduction over \mathbb{Z} and $0 < ht(\Delta) \leq \mathcal{B}$, has the asymptotic behavior:

$$a\mathcal{B}^{\frac{5}{6}} + b\mathcal{B}^{\frac{1}{2}} + c\mathcal{B}^{\frac{1}{3}} + o(\mathcal{B}^{\frac{1}{3}})$$

with main leading term $\mathcal{O}\left(\mathcal{B}^{\frac{5}{6}}\right)$, secondary term $\mathcal{O}\left(\mathcal{B}^{\frac{1}{2}}\right)$, tertiary term $\mathcal{O}\left(\mathcal{B}^{\frac{1}{3}}\right)$ and lower order terms.

Thank you :)

Thank you to the **organizers & everyone** for listening!